The Vector

[2] The Vector
The Vector: William Rowan Hamilton

By age 5, Latin, Greek, and Hebrew
By age 10, twelve languages including Persian, Arabic, Hindustani and Sanskrit.

William Rowan Hamilton, the inventor of the theory of quaternions...

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The Vector: Josiah Willard Gibbs

Started at Yale at 15
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Spent three years in Europe
Returned to be professor at Yale

Developed vector analysis as an alternative to quaternions.

His unpublished notes were passed around for twenty years.

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What is a vector?

- This is a 4-vector over $\mathbb{R}$:

  \[ [3.14159, 2.718281828, -1.0, 2.0] \]

- We will often use Python’s lists to represent vectors.

- Set of all 4-vectors over $\mathbb{R}$ is written $\mathbb{R}^4$.

- This notation might remind you of the notation $\mathbb{R}^D$: the set of functions from $D$ to $\mathbb{R}$. 
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Vectors are functions

Think of our 4-vector $[3.14159, 2.718281828, -1.0, 2.0]$ as the function

$$0 \mapsto 3.14159, \quad 1 \mapsto 2.718281828, \quad 2 \mapsto -1.0, \quad 3 \mapsto 2.0$$

$\mathbb{F}^d$ is notation for set of functions from $\{0, 1, 2, \ldots, d - 1\}$ to $\mathbb{F}$.

**Example:** $GF(2)^5$ is set of 5-element bit sequences, e.g. $[0,0,0,0,0]$, $[0,0,0,0,1]$, ...

Let $WORDS =$ set of all English words

In information retrieval, a document is represented ("bag of words" model) by a function $f : WORDS \to \mathbb{R}$ specifying, for each word, how many times it appears in the document.

We would refer to such a function as a WORDS-vector over $\mathbb{R}$

**Definition:** For a field $\mathbb{F}$ and a set $D$, a $D$-vector over $\mathbb{F}$ is a function from $D$ to $\mathbb{F}$. The set of such functions is written $\mathbb{F}^D$

For example, $\mathbb{R}^{WORDS}$
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Representation of vectors using Python dictionaries

We often use Python’s dictionaries to represent such functions, e.g.

\{0:3.14159, 1:2.718281828, 2:-1.0, 3:2.0\}

What about representing a WORDS-vector over \( \mathbb{R} \)?

For any single document, most words are not represented. They should be mapped to zero.

Our convention for representing vectors by dictionaries: we are allowed to omit key-value pairs when value is zero.

**Example:** “The rain in Spain falls mainly on the plain” would be represented by the dictionary

\{'on': 1, 'Spain': 1, 'in': 1, 'plain': 1, 'the': 2, 'mainly': 1, 'rain': 1, 'falls': 1\}
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A vector most of whose values are zero is called a \textit{sparse} vector. If no more than \( k \) of the entries are nonzero, we say the vector is \( k \)-\textit{sparse}. A \( k \)-sparse vector can be represented using space proportional to \( k \).

\textbf{Example:} when we represent a corpus of documents by WORD-vectors, the storage required is proportional to the total number of words in all documents.

Most signals acquired via physical sensors (images, sound, ...) are not exactly sparse. Later we study \textit{lossy compression}: making them sparse while preserving perceptual similarity.
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What can we represent with a vector?

- Document (for information retrieval)
- Binary string (for cryptography/information theory)
- Collection of attributes
  - Senate voting record
  - Demographic record of a consumer
  - Characteristics of cancer cells
- State of a system
  - Population distribution in the world
  - Number of copies of a virus in a computer network
  - State of a pseudorandom generator
  - State of *Lights Out*
- Probability distribution, e.g. \{1:1/6, 2:1/6, 3:1/6, 4:1/6, 5:1/6, 6:1/6\}
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What can we represent with a vector?

- **Image**
  
  \[
  \{(0,0): 0, (0,1): 0, (0,2): 0, (0,3): 0, \\
  (1,0): 32, (1,1): 32, (1,2): 32, (1,3): 32, \\
  (2,0): 64, (2,1): 64, (2,2): 64, (2,3): 64, \\
  (6,0): 192, (6,1): 192, (6,2): 192, (6,3): 192, \\
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What can we represent with a vector?

► Points
  ▶ Can interpret the 2-vector \([x, y]\) as a point in the plane.
  ▶ Can interpret 3-vectors as points in space, and so on.
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Vector addition: Translation and vector addition

With complex numbers, translation achieved by adding a complex number, e.g. 
\[ f(z) = z + (1 + 2i) \]

Let’s do the same thing with vectors...

**Definition of vector addition:**

\[ [u_1, u_2, \ldots, u_n] + [v_1, v_2, \ldots, v_n] = [u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n] \]

For 2-vectors represented in Python as 2-element lists, addition procedure is

```python
def add2(v, w):
    return [v[0]+w[0], v[1]+w[1]]
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**Quiz:** Suppose we represent $n$-vectors by $n$-element lists. Write a procedure `addn(v, w)` to compute the sum of two vectors so represented.
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**Answer:**

```python
def addn(v, w): return [v[i]+w[i] for i in range(len(v))]
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Vector addition: The zero vector

The $D$-vector whose entries are all zero is the zero vector, written $0_D$ or just $0$.

$$v + 0 = v$$
Vector addition: The zero vector

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$$v + 0 = v$$
Vector addition: Vector addition is associative and commutative

- **Associativity**
  \[(x + y) + z = x + (y + z)\]

- **Commutativity**
  \[x + y = y + x\]
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Like complex numbers in the plane, \( n \)-vectors over \( \mathbb{R} \) can be visualized as arrows in \( \mathbb{R}^n \).

The 2-vector \([3, 1.5]\) can be represented by an arrow with its tail at the origin and its head at \((3, 1.5)\).

or, equivalently, by an arrow whose tail is at \((-2, -1)\) and whose head is at \((1, 0.5)\).
Vector addition: Vectors as arrows

Like complex numbers in the plane, \( n \)-vectors over \( \mathbb{R} \) can be visualized as \textit{arrows} in \( \mathbb{R}^n \).

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Like complex numbers, addition of vectors over $\mathbb{R}$ can be visualized using arrows.

To add $\mathbf{u}$ and $\mathbf{v}$:

- place tail of $\mathbf{v}$’s arrow on head of $\mathbf{u}$’s arrow;
- draw a new arrow from tail of $\mathbf{u}$ to head of $\mathbf{v}$. 
Vector addition: Vectors as arrows

Like complex numbers, addition of vectors over \( \mathbb{R} \) can be visualized using arrows.

To add \( \mathbf{u} \) and \( \mathbf{v} \):

- place tail of \( \mathbf{v} \)'s arrow on head of \( \mathbf{u} \)'s arrow;
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Scalar-vector multiplication

With complex numbers, \textit{scaling} was multiplication by a real number \( f(z) = rz \)

For vectors,

- we refer to field elements as \textit{scalars};
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\[ \alpha v \]

Greek letters (e.g. \( \alpha, \beta, \gamma \)) denote scalars.
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Scalar-vector multiplication

**Definition:** Multiplying a vector \( \mathbf{v} \) by a scalar \( \alpha \) is defined as multiplying each entry of \( \mathbf{v} \) by \( \alpha \):

\[
\alpha [v_1, v_2, \ldots, v_n] = [\alpha v_1, \alpha v_2, \ldots, \alpha v_n]
\]

**Example:** \( 2 [5, 4, 10] = [2 \cdot 5, 2 \cdot 4, 2 \cdot 10] = [10, 8, 20] \)
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**Quiz:** Suppose we represent $n$-vectors by $n$-element lists. Write a procedure `scalar_vector_mult(alpha, v)` that multiplies the vector $v$ by the scalar $alpha$. 

```python
def scalar_vector_mult(alpha, v):
    return [alpha*x for x in v]
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Scalar-vector multiplication: Scaling arrows

An arrow representing the vector $[3, 1.5]$ is this:

and an arrow representing two times this vector is this:
Scalar-vector multiplication: Scaling arrows

An arrow representing the vector $[3, 1.5]$ is this:

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Scalar-vector multiplication: Associativity of scalar-vector multiplication

Associativity: $\alpha(\beta \mathbf{v}) = (\alpha \beta)\mathbf{v}$
Scalar-vector multiplication: Line segments through the origin

Consider scalar multiples of $\mathbf{v} = [3, 2]$: 
\{0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0\}

For each value of $\alpha$ in this set, 
$\alpha \mathbf{v}$ is shorter than $\mathbf{v}$ but in same direction.
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\( \alpha \mathbf{v} \) is shorter than \( \mathbf{v} \) but in same direction.
**Conclusion:** The set of points

$$\{ \alpha \mathbf{v} : \alpha \in \mathbb{R}, 0 \leq \alpha \leq 1 \}$$

forms the line segment between the origin and \( \mathbf{v} \)
Scalar-vector multiplication: Lines through the origin

What if we let $\alpha$ range over all real numbers?

- Scalars bigger than 1 give rise to somewhat larger copies
- Negative scalars give rise to vectors pointing in the opposite direction
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Combining vector addition and scalar multiplication

We want to describe the set of points forming an arbitrary line segment (not necessarily through the origin).

Idea: Use the idea of translation.

Start with line segment from $[0, 0]$ to $[3, 2]$:

$$\{\alpha [3, 2] : 0 \leq \alpha \leq 1\}$$

Translate it by adding $[0.5, 1]$ to every point:

$$\{[0.5, 1] + \alpha [3, 2] : 0 \leq \alpha \leq 1\}$$

Get line segment from $[0, 0]+[0.5, 1]$ to $[3, 2]+[0.5, 1]$. 
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Get line segment from [0, 0] + [0.5, 1] to [3, 2] + [0.5, 1]
Combining vector addition and scalar multiplication: Distributive laws for scalar-vector multiplication and vector addition

*Scalar-vector multiplication distributes over vector addition:*

\[
\alpha (u + v) = \alpha u + \alpha v
\]

Example:

- On the one hand,
  \[
  2 ([1, 2, 3] + [3, 4, 4]) = 2 [4, 6, 7] = [8, 12, 14]
  \]

- On the other hand,
  \[
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Combining vector addition and scalar multiplication: First look at convex combinations

Set of points making up the the $[0.5, 1]$-to-$[3.5, 3]$ segment:

$$\{ \alpha [3, 2] + [0.5, 1] : \alpha \in \mathbb{R}, 0 \leq \alpha \leq 1 \}$$

Not symmetric with respect to endpoints 😞
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Use distributivity:
\[
\alpha [3, 2] + [0.5, 1] = \alpha ([3.5, 3] - [0.5, 1]) + [0.5, 1]
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\]

where \( \beta = 1 - \alpha \)
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\end{align*}
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where \( \beta = 1 - \alpha \)

New formulation:

\[ \{ \alpha [3.5, 3] + \beta [0.5, 1] : \alpha, \beta \in \mathbb{R}, \alpha, \beta \geq 0, \alpha + \beta = 1 \} \]
Combining vector addition and scalar multiplication: First look at convex combinations

Set of points making up the the $[0.5, 1]$-to-$[3.5, 3]$ segment:

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$$\alpha [3, 2] + [0.5, 1] = α ([3.5, 3] − [0.5, 1]) + [0.5, 1]$$
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$$= \alpha [3.5, 3] + (1 − \alpha) [0.5, 1]$$
$$= \alpha [3.5, 3] + β [0.5, 1]$$

where $β = 1 − α$

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$$\{α [3.5, 3] + β [0.5, 1] : α, β ∈ \mathbb{R}, α, β ≥ 0, α + β = 1\}$$

Symmetric with respect to endpoints 😊
Combining vector addition and scalar multiplication: First look at convex combinations

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An expression of the form

\[ \alpha \mathbf{u} + \beta \mathbf{v} \]

where \(0 \leq \alpha \leq 1, 0 \leq \beta \leq 1, \) and \(\alpha + \beta = 1\) is called a convex combination of \(\mathbf{u}\) and \(\mathbf{v}\)

The \(\mathbf{u}\)-to-\(\mathbf{v}\) line segment consists of the set of convex combinations of \(\mathbf{u}\) and \(\mathbf{v}\).
Combining vector addition and scalar multiplication: First look at convex combinations

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Combining vector addition and scalar multiplication: First look at convex combinations

\[ \mathbf{u} = \text{[image 1]} \quad \text{and} \quad \mathbf{v} = \text{[image 2]} \]

Use scalars \( \alpha = \frac{1}{2} \) and \( \beta = \frac{1}{2} \):

\[
\frac{1}{2} \mathbf{u} + \frac{1}{2} \mathbf{v} =
\]
Combining vector addition and scalar multiplication: First look at convex combinations

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“Line segment” between two faces:
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Combining vector addition and scalar multiplication: First look at convex combinations
Combining vector addition and scalar multiplication: First look at affine combinations

Infinite line through [0.5, 1] and [3.5, 3]?

Our formulation so far 😞:

\[ \{ [0.5, 1] + \alpha [3, 2] : \alpha \in \mathbb{R} \} \]

Nicer formulation 😊:

\[ \{ \alpha [3.5, 3] + \beta [0.5, 1] : \alpha \in \mathbb{R}, \beta \in \mathbb{R}, \alpha + \beta = 1 \} \]

An expression of the form \( \alpha \mathbf{u} + \beta \mathbf{v} \) where \( \alpha + \beta = 1 \) is called an affine combination of \( \mathbf{u} \) and \( \mathbf{v} \).

The line through \( \mathbf{u} \) and \( \mathbf{v} \) consists of the set of affine combinations of \( \mathbf{u} \) and \( \mathbf{v} \).
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Vectors over $GF(2)$

Addition of vectors over $GF(2)$:

\[
\begin{array}{cccccc}
1 & 1 & 1 & 1 & 1 & 1 \\
+ & 1 & 0 & 1 & 0 & 1 \\
\hline
0 & 1 & 0 & 1 & 0 & 0 \\
\end{array}
\]

For brevity, in doing $GF(2)$, we often write 1101 instead of $[1,1,0,1]$.

Example: Over $GF(2)$, what is 1101 + 0111?
Answer: 1010
Vectors over $GF(2)$

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Vectors over $GF(2)$: Perfect secrecy

Represent encryption of $n$ bits by addition of $n$-vectors over $GF(2)$.

**Example:**

Alice and Bob agree on the following 10-vector as a key:

$$k = [0, 1, 1, 0, 1, 0, 0, 0, 1]$$

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If the key is chosen according to the uniform distribution, encryption by addition of vectors over \( GF(2) \) achieves \textit{perfect secrecy}. For each plaintext \( p \), the function that maps the key to the cyphertext

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- What can Alice learn without Bob?
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RSA just introduced a product based on this idea:

- Split each password into two parts.
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Vectors over $GF(2)$: *Lights Out*

- *input*: Configuration of lights
- *output*: Which buttons to press in order to turn off all lights?

**Computational Problem:** Solve an instance of *Lights Out*

Represent state using $\text{range}(5) \times \text{range}(5)$-vector over $GF(2)$.

**Example state vector:**

Represent each button as a vector (with ones in positions that the button toggles)

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![State Vector](image.png)

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Look at $3 \times 3$ case.

\[
\begin{array}{c}
\text{state} \\
\begin{array}{c}
. \\
. \\
. \\
\end{array} \\
\end{array} + \\
\begin{array}{c}
\text{move} \\
\begin{array}{c}
. \\
. \\
. \\
\end{array} \\
\end{array} = \\
\begin{array}{c}
\text{new state} \\
\begin{array}{c}
. \\
. \\
. \\
\end{array} \\
\end{array}
\]

\[
\begin{array}{c}
\text{state} \\
\begin{array}{c}
\cdot \\
\cdot \\
\cdot \\
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\cdot \\
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Vectors over $GF(2)$: *Lights Out*

Look at $3 \times 3$ case.

- State move new state
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Vectors over $GF(2)$: *Lights Out*

Look at $3 \times 3$ case.

\[
\begin{align*}
\text{state} & \quad + & \quad \text{move} & \quad = & \quad \text{new state} \\
\begin{array}{ccc}
\bullet & \bullet & \bullet \\
\bullet & & \bullet \\
\end{array} & \quad + & \quad \begin{array}{ccc}
\bullet & \bullet & \bullet \\
\bullet & & \bullet \\
\end{array} & \quad = & \quad \begin{array}{ccc}
\bullet & \bullet & \bullet \\
\bullet & & \bullet \\
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\bullet & \bullet & \bullet \\
\bullet & & \bullet \\
\end{array} & \quad = & \quad \begin{array}{ccc}
\bullet & \bullet & \bullet \\
\bullet & & \bullet \\
\end{array} \\
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\begin{array}{ccc}
\bullet & \bullet & \bullet \\
\bullet & & \bullet \\
\end{array} & \quad + & \quad \begin{array}{ccc}
\bullet & \bullet & \bullet \\
\bullet & & \bullet \\
\end{array} & \quad = & \quad \begin{array}{ccc}
\bullet & \bullet & \bullet \\
\bullet & & \bullet \\
\end{array}
\end{align*}
\]
Vectors over $GF(2)$: Lights Out

Button vectors for $3 \times 3$:

![Button vectors for 3x3](image)

Computational Problem: Which sequence of button vectors sum to $s$?
Vectors over $GF(2)$: *Lights Out*

**Computational Problem:** Which sequence of button vectors sum to $s$?

Observations:

- By commutative property of vector addition, order doesn’t matter.
- A button vector occurring twice cancels out.

Replace Computational Problem with: Which *set* of button vectors sum to $s$?
Vectors over $GF(2)$: *Lights Out*

**Computational Problem:** Which sequence of button vectors sum to $s$?

Observations:

- By commutative property of vector addition, order doesn’t matter.
- A button vector occurring twice cancels out.

Replace Computational Problem with: Which *set* of button vectors sum to $s$?
Vectors over $GF(2)$: *Lights Out*

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Replace our original Computational Problem with a more general one:

Solve an instance of *Lights Out* $\Rightarrow$ Which set of button vectors sum to $s$?

$\Rightarrow$ Find subset of $GF(2)$ vectors $v_1, \ldots, v_n$ whose sum equals $s$
Vectors over $GF(2)$: *Lights Out*

Button vectors for $2 \times 2$ version:

$$
\begin{array}{cccc}
\bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet \\
\end{array}
$$

where the black dots represent ones.

**Quiz:** Find the subset of the button vectors whose sum is

$$
\begin{array}{ccc}
\bullet & \bullet \\
\bullet & \bullet \\
\end{array}
$$
Vectors over $GF(2)$: *Lights Out*

Button vectors for $2 \times 2$ version:

\[
\begin{array}{cccc}
\cdot \cdot & \cdot \cdot & \cdot \cdot & \cdot \cdot \\
\cdot \cdot & \cdot \cdot & \cdot \cdot & \cdot \cdot 
\end{array}
\]

where the black dots represent ones.

**Quiz:** Find the subset of the button vectors whose sum is

\[
\begin{array}{c}
\cdot \\
\cdot 
\end{array}
\]

**Answer:**

\[
\begin{array}{cccc}
\cdot \cdot & \cdot \cdot & \cdot \cdot & \cdot \cdot \\
\cdot \cdot & \cdot \cdot & \cdot \cdot & \cdot \cdot 
\end{array}
= \begin{array}{c}
\cdot \\
\cdot 
\end{array} + \begin{array}{c}
\cdot \\
\cdot 
\end{array}
\]
**Dot-product**

*Dot-product* of two $D$-vectors is sum of product of corresponding entries:

$$
\mathbf{u} \cdot \mathbf{v} = \sum_{k \in D} u[k] \cdot v[k]
$$

**Example:** For traditional vectors $\mathbf{u} = [u_1, \ldots, u_n]$ and $\mathbf{v} = [v_1, \ldots, v_n]$,

$$
\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + \cdots + u_nv_n
$$

Output is a scalar, not a vector.

Dot-product sometimes called *scalar product*. 
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Dot-product sometimes called *scalar product*. 
Dot-product

**Example:** Dot-product of \([1, 1, 1, 1, 1]\) and \([10, 20, 0, 40, -100]\):

\[
\begin{array}{cccccc}
1 & 1 & 1 & 1 & 1 & 1 \\
\cdot & 10 & 20 & 0 & 40 & -100 \\
10 + 20 + 0 + 40 + (-100) &=& -30
\end{array}
\]
Example: Dot-product of \([1, 1, 1, 1, 1]\) and \([10, 20, 0, 40, -100]\):

\[
\begin{array}{cccccc}
1 & 1 & 1 & 1 & 1 & 1 \\
10 & 20 & 0 & 40 & -100 \\
\hline
10 & + & 20 & + & 0 & + & 40 & + & (-100) &= & -30
\end{array}
\]
Quiz: Dot-product

**Quiz:** Write a procedure `list_dot(u, v)` with the following spec:

- **input:** equal-length lists `u` and `v` of field elements
- **output:** the dot-product of `u` and `v` interpreted as vectors

**Hint:** Use the `sum(·)` procedure together with a list comprehension.

```python
def list_dot(u, v):
    return sum([u[i]*v[i] for i in range(len(u))])
```

```python
def list_dot(u, v):
    return sum([a*b for (a,b) in zip(u,v)])
```
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**Answer**:

```python
def list_dot(u, v):
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or

```python
def list_dot(u, v):
    return sum([a*b for (a,b) in zip(u,v)])
```
Dot-product: Total cost or benefit

Suppose \( D \) consists of four main ingredients of beer:

\[ D = \{ \text{malt, hops, yeast, water} \} \]

A \textit{cost} vector maps each food to a price per unit amount:

\[ \text{cost} = \{ \text{hops : } $2.50/\text{ounce}, \text{malt : } $1.50/\text{pound}, \text{water : } $0.06/\text{gallon}, \text{yeast : } $0.45/\text{g} \} \]

A \textit{quantity} vector maps each food to an amount (e.g. measured in pounds).

\[ \text{quantity} = \{ \text{hops:6 oz, malt:14 pounds, water:7 gallons, yeast:11 grams} \} \]

The total cost is the dot-product of \textit{cost} with \textit{quantity}:

\[ \text{cost} \cdot \text{quantity} = $2.50 \cdot 6 + $1.50 \cdot 14 + $0.006 \cdot 7 + $0.45 \cdot 11 = $40.992 \]

A \textit{value} vector maps each food to its caloric content per pound:

\[ \text{value} = \{ \text{hops : 0, malt : 960, water : 0, yeast : 3.25} \} \]

The total calories represented by a pint of beer is the dot-product of \textit{value} with \textit{quantity}:
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**Dot-product: Linear equations**

**Example:** A sensor node consists of hardware components, e.g.
- CPU
- radio
- temperature sensor
- memory

Battery-driven and remotely located so we care about energy usage.

Suppose we know the power consumption for each hardware component. Represent it as a $D$-vector with $D = \{\text{radio, sensor, memory, CPU}\}$

\[
\text{rate} = \text{Vec}(D, \{\text{memory} : 0.06\text{W}, \text{radio} : 0.06\text{W}, \text{sensor} : 0.004\text{W}, \text{CPU} : 0.0025\text{W}\})
\]

Have a test period during which we know how long each component was working. Represent as another $D$ vector:

\[
\text{duration} = \text{Vec}(D, \{\text{memory} : 1.0\text{s}, \text{radio} : 0.2\text{s}, \text{sensor} : 0.5\text{s}, \text{CPU} : 1.0\text{s}\})
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Total energy consumed (in Joules):

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\text{duration} \cdot \text{rate}
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Dot-product: Linear equations

*Turns out:* We can only measure *total energy consumed by the sensor node over a period*

**Goal:** calculate rate of energy consumption of each hardware component.

**Challenge:** Cannot simply turn on memory without turning on CPU.

**Idea:**

- Run several tests on sensor node in which we measure total energy consumption
- In each test period, we know the duration each hardware component is turned on.

For example,

- \( \text{duration}_1 = \{\text{radio} : 0.2s, \text{sensor} : 0.5s, \text{memory} : 1.0s, \text{CPU} : 1.0s\} \)
- \( \text{duration}_2 = \{\text{radio} : 0s, \text{sensor} : 0.1s, \text{memory} : 0.2s, \text{CPU} : 0.5s\} \)
- \( \text{duration}_3 = \{\text{radio} : 0.4s, \text{sensor} : 0s, \text{memory} : 0.2s, \text{CPU} : 1.0s\} \)

- In each test period, we know the total energy consumed:
  \( \beta_1 = 1, \beta_2 = 0.75, \beta_3 = .6 \)

- Use data to calculate current for each hardware component
Dot-product: Linear equations

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A *linear equation* is an equation of the form

\[ \mathbf{a} \cdot \mathbf{x} = \beta \]

where \( \mathbf{a} \) is a vector, \( \beta \) is a scalar, and \( \mathbf{x} \) is a vector of variables.

In sensor-node problem, we have linear equations of the form

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Questions:

- Can we find numbers for the entries of \( \text{rate} \) such that the equations hold?
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Dot-product: Linear equations

A \emph{linear equation} is an equation of the form

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More general questions:

- Is there an algorithm for solving a system of linear equations?

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    \vdots & \quad \vdots \\
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- How can we know whether there is only one solution?

- What if our data are slightly inaccurate?

These questions motivate much of what is coming in future weeks.
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Dot-product: Measuring similarity: Comparing voting records

Can use dot-product to measure similarity between vectors.

**Upcoming lab:**

- Represent each senator’s voting record as a vector:
  
  \[ [+1, +1, 0, -1] \]

  +1 = *In favor*, 0 = *not voting*, -1 = *against*

- Dot-product \([+1, +1, 0, -1] \cdot [-1, -1, -1, +1] \)
  
  very positive if the two senators tend to agree,
  very negative if two voting records tend to disagree.
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![Sampling a Waveform](image)

To compare two equal-length sequences of samples, use dot-product:

\[ \sum_{i=1}^{n} u[i] \cdot v[i]. \]

- Term *i* in this sum is positive if \( u[i] \) and \( v[i] \) have the same sign, and negative if they have opposite signs.
- The greater the agreement, the greater the value of the dot-product.
**Dot-product: Measuring similarity: Comparing audio segments**

**Back to needle-in-a-haystack:**
If you suspect you know where the needle is...

<table>
<thead>
<tr>
<th>5</th>
<th>-6</th>
<th>9</th>
<th>-9</th>
<th>-5</th>
<th>-9</th>
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<th>5</th>
<th>-8</th>
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<th>6</th>
<th>-2</th>
<th>-4</th>
<th>-9</th>
<th>-1</th>
<th>-1</th>
<th>-9</th>
<th>-3</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>7</td>
<td>4</td>
<td>-3</td>
<td>0</td>
<td>-1</td>
<td>-6</td>
<td>4</td>
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Dot-product: Measuring similarity: Comparing audio segments

If you don’t have any idea where to find the needle, compute lots of dot-products!
Dot-product: Measuring similarity: Comparing audio segments

Seems like a lot of dot-products—too much computation—but there is a shortcut...
The Fast Fourier Transform.
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Seems like a lot of dot-products—too much computation—but there is a shortcut... The *Fast Fourier Transform*. 
Dot-product: Vectors over $GF(2)$

Consider the dot-product of 11111 and 10101:

\[
\begin{array}{cccccc}
1 & 1 & 1 & 1 & 1 & 1 \\
\times & 1 & 0 & 1 & 0 & 1 \\
\hline
1 & + & 0 & + & 1 & + & 0 & + & 1 & = & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
\times & 0 & 0 & 1 & 0 & 1 \\
\hline
0 & + & 0 & + & 1 & + & 0 & + & 1 & = & 0
\end{array}
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Dot-product: Simple authentication scheme

- Usual way of logging into a computer with a password is subject to hacking by an eavesdropper.

- **Alternative**: Challenge-response system
  - Computer asks a question about the password.
  - Human sends the answer.
  - Repeat a few times before human is considered authenticated.

  Potentially safe against an eavesdropper since probably next time will involve different questions.

- Simple challenge-response scheme based on dot-product of vectors over $GF(2)$:
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Example: Password is $\hat{x} = 10111$.

Computer sends $a_1 = 01011$ to Human.

Human computes dot-product

\[ a_1 \cdot \hat{x} : \]

\[
\begin{array}{cccccc}
0 & 1 & 0 & 1 & 1 \\
1 & 0 & 1 & 1 & 1 \\
\end{array}
\]

\[
0 + 0 + 0 + 1 + 1 = 0
\]

and sends $\beta_1 = 0$ to Computer.
Dot-product: Attacking simple authentication scheme

How can an eavesdropper Eve cheat?

- She observes a sequence of challenge vectors $a_1, a_2, \ldots, a_m$ and the corresponding response bits $\beta_1, \beta_2, \ldots, \beta_m$.

- Can she find the password?

She knows the password must satisfy the linear equations

\[
\begin{align*}
a_1 \cdot x &= \beta_1 \\
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Questions:

- How many solutions?
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Dot-product: Attacking simple authentication scheme

Another way to cheat?

Can Eve derive a challenge for which she knows the response?

Algebraic properties of dot-product:

- **Commutativity**: $v \cdot x = x \cdot v$
- **Homogeneity**: $(\alpha u) \cdot v = \alpha (u \cdot v)$
- **Distributive law**: $(v_1 + v_2) \cdot x = v_1 \cdot x + v_2 \cdot x$

Example: Eve observes

- challenge 01011, response 0
- challenge 11110, response 1

\[
(01011 + 11110) \cdot x = 01011 \cdot x + 11110 \cdot x \\
= 0 + 1 \\
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For challenge 01011 + 11110, Eve can derive right response.
More generally, if a vector satisfies equations

\[
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then what other equations does the vector satisfy?
Answer will come later.
Dictionary-based representations of vectors

- A vector is a function from some domain $D$ to a field
- Can represent such a function in Python by a dictionary.
- It’s convenient to define a Python class `Vec` with two instance variables (fields):
  - $f$, the function, represented by a Python dictionary, and
  - $D$, the domain of the function, represented by a Python set.
- We adopt the convention in which entries with value zero may be omitted from the dictionary $f$

(Simplified) class definition:

```python
class Vec:
    def __init__(self, labels, function):
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Can then create an instance:

>>> Vec({'A','B','C'}, {'A':1})

- First argument is assigned to $D$ field.
- Second argument is assigned to $f$ field.
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Can then create an instance:

>>> Vec({'A','B','C'}, {'A':1})

- First argument is assigned to D field.
- Second argument is assigned to f field.
Dictionary-based representations of vectors

(Simplified) class definition:

```python
class Vec:
    def __init__(self, labels, function):
        self.D = labels
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```

Can then create an instance:

```python
>>> Vec({'A','B','C'}, {'A':1})
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- First argument is assigned to `D` field.
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Dictionary-based representations of vectors

Can assign an instance to a variable:

>>> v=Vec({'A','B','C'}, {'A':1.})

and subsequently access the two fields of v, e.g.:

>>> for d in v.D:
...   if d in v.f:
...     print(v.f[d])
...  
1.0
Dictionary-based representations of vectors

**Quiz:** Write a procedure `zero_vec(D)` with the following spec:

- **input:** a set $D$
- **output:** an instance of `Vec` representing a $D$-vector all of whose entries have value zero

**Answer:**

```python
def zero_vec(D):
    return Vec(D, {})
```

or

```python
def zero_vec(D):
    return Vec(D, {d:0 for d in D})
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Setter:

```python
def setitem(v, d, val): v.f[d] = val
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- Second argument should be member of v.D.
- Third argument should be an element of the field.
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Example:

```python
>>> setitem(v, 'B', 2.)
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Dictionary-based representations of vectors: Setter and getter

**Quiz:** Write a procedure `getitem(v, d)` with the following spec:

- **input:** an instance `v` of `Vec`, and an element `d` of the set `v.D`
- **output:** the value of entry `d` of `v`

**Answer:**
```python
def getitem(v, d):
    if d in v.f:
        return v.f[d]
    else:
        return 0
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*Why is `getitem(v, d): return v.f[d]` not enough?*

**Sparsity convention**
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Using Vec

You will write the bodies of named procedures such as `setitem(v, d, val)` and `add(u,v)` and `scalar_mul(v, alpha)`.

However, in actually using Vecs in other code, you must use operators instead of named procedures, e.g.

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Assertions in Vec

For each procedure you write, we will provide the stub of the procedure, e.g. for add(u, v), we provide the stub

```python
def add(u, v):
    "Returns the sum of the two vectors"
    assert u.D == v.D
    pass
```

The first line in the body is a documentation string, basically a comment.

The second line is an assertion. It asserts that the two arguments u and v must have equal domains. If the procedure is called with arguments that violate this, Python reports an error.

The assertion is there to remind us that two vectors can be added only if they have the same domain.

Please keep the assertions in your vec code while using it for this course.
Assertions in Vec

For each procedure you write, we will provide the stub of the procedure, e.g. for `add(u, v)`, we provide the stub

```python
def add(u, v):
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Please keep the assertions in your vec code while using it for this course.
Testing Vec with doctests

We have provided tests in the docstrings:

```python
def getitem(v,k):
    """
    Return the value of entry d in v.
    >>> v = Vec({‘a’,’b’,’c’, ’d’},
               {’a’:2,’c’:1,’d’:3})
    >>> v[’d’]
    3
    >>> v[’b’]
    0
    """
    pass
```

Tests show interactions with Python assuming correct implementation.

You can copy from the file and paste into your Python session.

You can also run all the tests at once from the console (outside the Python interpreter) using the following command:

```
python3 -m doctest vec.py
```

This will run the tests given in vec.py and will print messages about any discrepancies that arise. If your code passes the tests, nothing will be printed.
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    >>> v['d']
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    0
    """
    pass
```

Tests show interactions with Python assuming correct implementation.

You can copy from the file and paste into your Python session.

You can also run all the tests at once from the console (outside the Python interpreter) using the following command:

```bash
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    3
    
    >>> v['b']
    0
    
    pass

Tests show interactions with Python assuming correct implementation.

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grep -m doctest vec.py

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**list2vec**

The *Vec* class is useful for representing vectors but is not the only useful representation. We sometimes represent vectors by lists.

A list \( L \) can be viewed as a function from \( \{0, 1, 2, \ldots, \text{len}(L) - 1\} \), so it is easy to convert between list-based and dictionary-based representations.

**Quiz:** Write a procedure `list2vec(L)` with the following spec:

- **input:** a list \( L \) of field elements
- **output:** an instance \( v \) of *Vec* with domain \( \{0, 1, 2, \ldots, \text{len}(L) - 1\} \) such that \( v[i] = L[i] \) for each integer \( i \) in the domain

```python
def list2vec(L):
    return Vec(set(range(len(L))), {k:x for k,x in enumerate(L)})

# or
def list2vec(L):
    return Vec(set(range(len(L))), {k:L[k] for k in range(len(L))})
```
list2vec

The Vec class is useful for representing vectors but is not the only useful representation. We sometimes represent vectors by lists. A list $L$ can be viewed as a function from $\{0, 1, 2, \ldots, \text{len}(L) - 1\}$, so it is easy to convert between list-based and dictionary-based representations.

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    return Vec(set(range(len(L))), {k:L[k] for k in range(len(L))})
```
The vecutil module

The procedures `zero_vec(D)` and `list2vec(L)` are defined in the file `vecutil.py`, which we provide.
Solving a triangular system of linear equations

How to find solution to this linear system?

\[
\begin{align*}
[1, 0.5, -2, 4] \cdot x &= -8 \\
[0, 3, 3, 2] \cdot x &= 3 \\
[0, 0, 1, 5] \cdot x &= -4 \\
[0, 0, 0, 2] \cdot x &= 6
\end{align*}
\]

Write \( x = [x_1, x_2, x_3, x_4] \).
System becomes

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\begin{align*}
1x_1 &+ 0.5x_2 - 2x_3 + 4x_4 = -8 \\
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Solving a triangular system of linear equations: Backward substitution

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**Solution strategy:**

- Solve for \(x_4\) using fourth equation.
- Plug value for \(x_4\) into third equations and solve for \(x_3\).
- Plug values for \(x_4\) and \(x_3\) into second equation and solve for \(x_2\).
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\[ 2x_4 = 6 \]
Solving a triangular system of linear equations: Backward substitution

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\end{align*}
\]

so \( x_4 = 6/2 = 3 \)
Solving a triangular system of linear equations: Backward substitution

\[\begin{align*}
1x_1 &+ 0.5x_2 - 2x_3 + 4x_4 = -8 \\
3x_2 &+ 3x_3 + 2x_4 = 3 \\
1x_3 &+ 5x_4 = -4 \\
2x_4 &= 6
\end{align*}\]

so \[x_4 = \frac{6}{2} = 3\]

\[\begin{align*}
1x_3 &= -4 - 5x_4 = -4 - 5(3) = -19
\end{align*}\]
Solving a triangular system of linear equations: Backward substitution

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\begin{align*}
1x_1 & + 0.5x_2 - 2x_3 + 4x_4 = -8 \\
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2x_4 & = 6
\end{align*}
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so \( x_4 = 6/2 = 3 \)

\[
\begin{align*}
1x_3 & = -4 - 5x_4 = -4 - 5(3) = -19 \\
so \quad x_3 & = -19/1 = -19
\end{align*}
\]
Solving a triangular system of linear equations: Backward substitution

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\begin{align*}
1x_1 &+ 0.5x_2 - 2x_3 + 4x_4 = -8 \\
3x_2 &+ 3x_3 + 2x_4 = 3 \\
1x_3 &+ 5x_4 = -4 \\
2x_4 &= 6
\end{align*}
\]

\[
\begin{align*}
2x_4 &= 6 \\
\text{so } x_4 &= 6/2 = 3
\end{align*}
\]

\[
\begin{align*}
1x_3 &= -4 - 5x_4 = -4 - 5(3) = -19 \\
\text{so } x_3 &= -19/1 = -19
\end{align*}
\]

\[
\begin{align*}
3x_2 &= 3 - 3x_3 - 2x_4 = 3 - 2(3) - 3(-19) = 54
\end{align*}
\]
Solving a triangular system of linear equations: Backward substitution

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\begin{align*}
1x_1 & + 0.5x_2 - 2x_3 + 4x_4 = -8 \\
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so \ x_3 & = -19/1 = -19
\end{align*}
\]

so \( x_3 = -19/1 = -19 \)

\[
\begin{align*}
3x_2 & = 3 - 3x_3 - 2x_4 = 3 - 2(3) - 3(-19) = 54 \\
so \ x_2 & = 54/3 = 18
\end{align*}
\]
Solving a triangular system of linear equations: Backward substitution

\[
\begin{align*}
1x_1 & + 0.5x_2 - 2x_3 + 4x_4 = -8 \\
3x_2 & + 3x_3 + 2x_4 = 3 \\
1x_3 & + 5x_4 = -4 \\
2x_4 & = 6
\end{align*}
\]

\[
\begin{align*}
x_4 & = 6/2 = 3 \\
x_3 & = -19/1 = -19 \\
x_2 & = 54/3 = 18 \\
x_1 & = -67
\end{align*}
\]
Solving a triangular system of linear equations: Backward substitution

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\begin{align*}
3x_2 & = 3 - 3x_3 - 2x_4 = 3 - 2(3) - 3(-19) = 54 \\
so \quad x_2 & = 54/3 = 18
\end{align*}
\]

\[
\begin{align*}
1x_1 & = -8 - 0.5x_2 + 2x_3 - 4x_4 = -8 - 4(3) + 2(-19) - 0.5(18) = -67 \\
so \quad x_1 & = -67/1 = -67
\end{align*}
\]
Solving a triangular system of linear equations: Backward substitution

**Quiz:** Solve the following system by hand:

\[
\begin{align*}
2x_1 + 3x_2 - 4x_3 &= 10 \\
1x_2 + 2x_3 &= 3 \\
5x_3 &= 15
\end{align*}
\]
Solving a triangular system of linear equations: Backward substitution

**Quiz:** Solve the following system by hand:

\[
\begin{align*}
2x_1 + 3x_2 - 4x_3 &= 10 \\
1x_2 + 2x_3 &= 3 \\
5x_3 &= 15
\end{align*}
\]

**Answer:**

\[
\begin{align*}
x_3 &= 15/5 = 3 \\
x_2 &= 3 - 2x_3 = -3 \\
x_1 &= (10 + 4x_3 - 3x_2)/2 = (10 + 12 + 9)/2 = 31/2
\end{align*}
\]
Solving a triangular system of linear equations: Backward substitution

Hack to implement backward substitution using vectors:

- Initialize vector $x$ to zero vector.
- Procedure will populate $x$ entry by entry.
- When it is time to populate $x_i$, entries $x_{i+1}, x_{i+2}, \ldots, x_n$ will be populated, and other entries will be zero.
- Therefore can use dot-product:
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Solving a triangular system of linear equations: Backward substitution

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Our code only works when vectors in rowlist have domain $D = \{0, 1, 2, \ldots, n-1\}$.

For arbitrary domains, need to specify an ordering for which system is “triangular”:

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