[6] Dimension

The size of a basis

Key fact for this week: all bases for a vector space have the same size.

We use this as the "basis" for answering many pending questions.

Morphing Lemma

Morphing Lemma: Suppose S is a set of vectors, and B is a linearly independent set of vectors in Span S. Then $|S| \geq |B|$.

Before we prove it—what good is this lemma?

Theorem: Any basis for $\mathcal{V}$ is a smallest generating set for $\mathcal{V}$.

Proof: Let S be a smallest generating set for $\mathcal{V}$. Let B be a basis for $\mathcal{V}$. Then B is a linearly independent set of vectors in Span S. By the Morphing Lemma, B is no bigger than S, so B is also a smallest generating set.

Theorem: All bases for a vector space $\mathcal{V}$ have the same size.

Proof: They are all smallest generating sets.

Proof of the Morphing Lemma

Morphing Lemma: Suppose S is a set of vectors, and B is a linearly independent set of vectors in Span S. Then $|S| \geq |B|$.

Proof outline: modify S step by step, introducing vectors of B one by one, without increasing the size.

How? Using the Exchange Lemma....

Review of Exchange Lemma

Exchange Lemma: Suppose S is a set of vectors and A is a subset of S. Suppose $\mathbf{z}$ is a vector in Span S such that $A \cup \{\mathbf{z}\}$ is linearly independent.
Then there is a vector $\mathbf{w} \in S - A$ such that

$$\text{Span } S = \text{Span } (S \cup \{\mathbf{z}\} - \{\mathbf{w}\})$$

Proof of the Morphing Lemma

Let $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$. Define $S_0 = S$.
Prove by induction on $k \leq n$ that there is a generating set S_k of Span S that contains $\mathbf{b}_1, \ldots, \mathbf{b}_k$ and has size $|S|$.

Base case: $k = 0$ is trivial.

To go from S_{k-1} to S_k: use the Exchange Lemma.

- $A_k = \{\mathbf{b}_1, \ldots, \mathbf{b}_{k-1}\}$ and $\mathbf{z} = \mathbf{b}_k$

Exchange Lemma $\Rightarrow$ there is a vector $\mathbf{w}$ in S_{k-1} such that

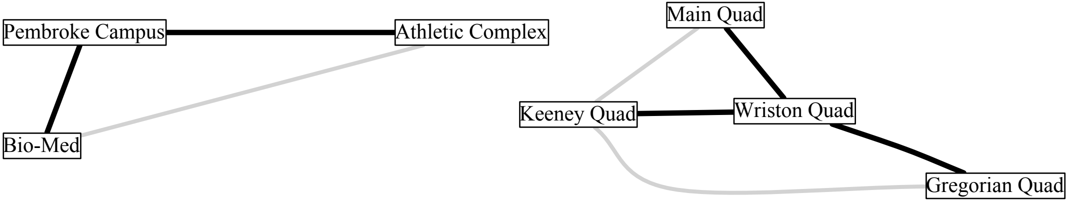
$$\text{Span } (S_{k-1} \cup \{\mathbf{b}_k\} - \{\mathbf{w}\}) = \text{Span } S_{k-1}$$

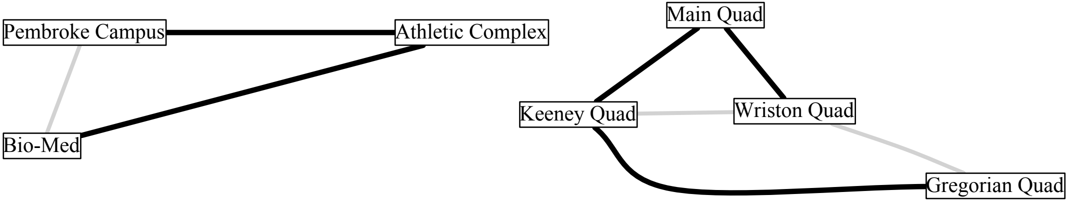
Set $S_k = S_{k-1} \cup \{\mathbf{b}_k\} - \{\mathbf{w}\}$.

QED

This induction proof is an algorithm.

Morphing from one spanning forest to another





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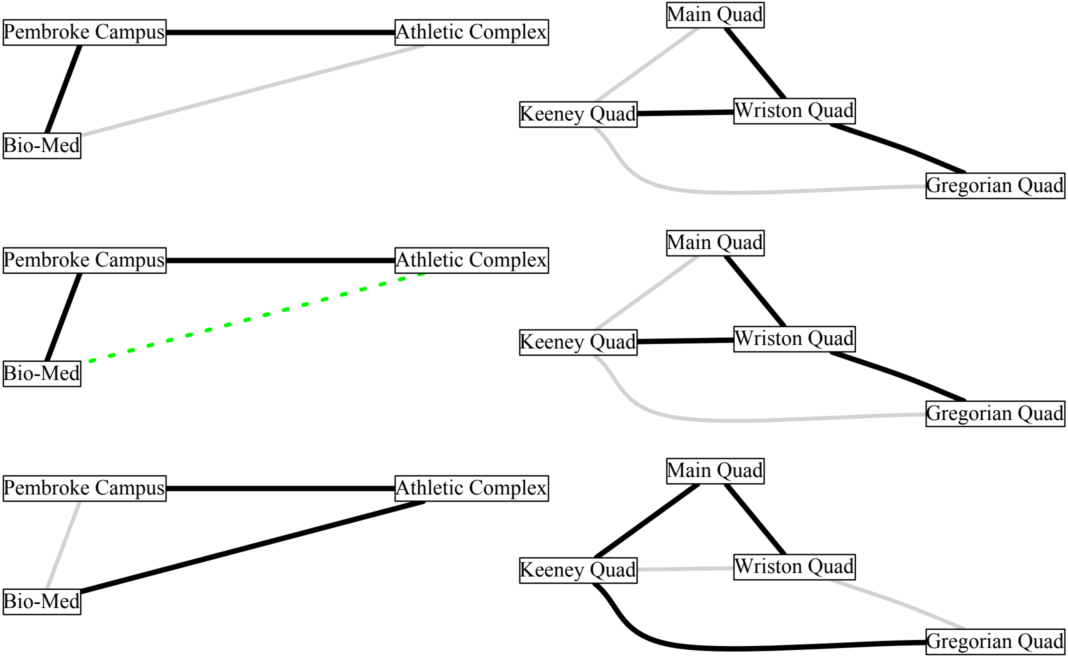

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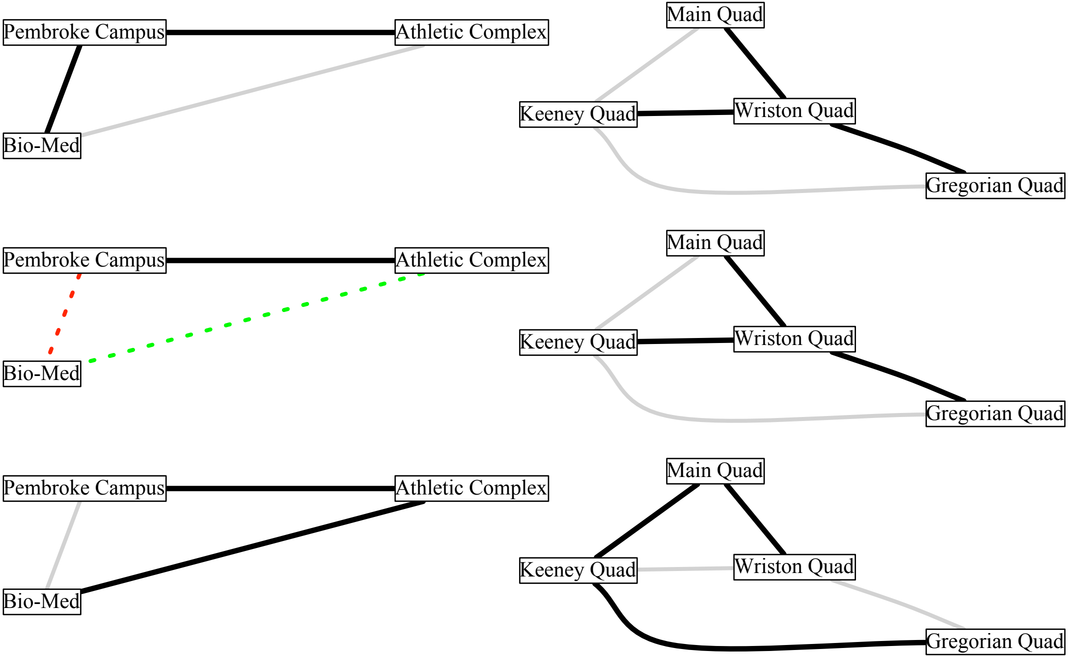

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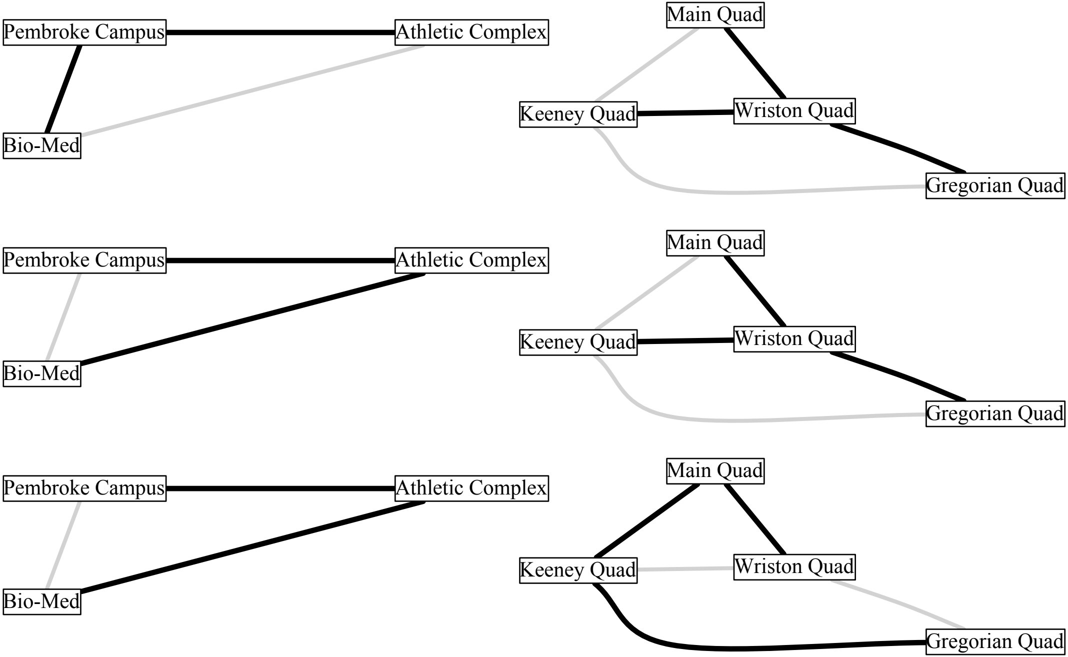

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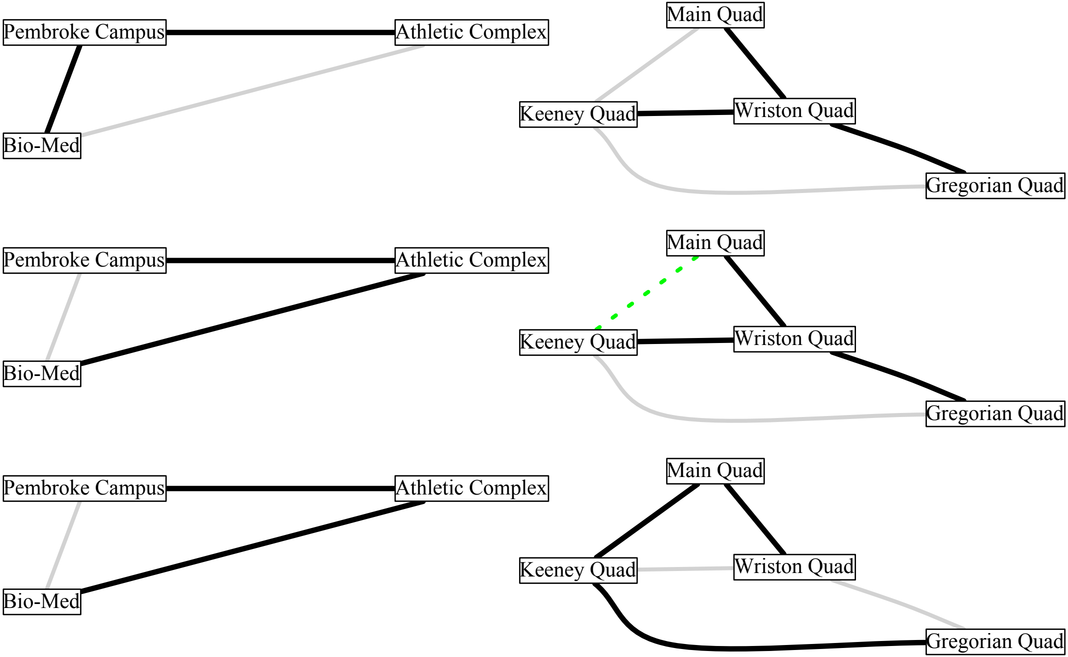

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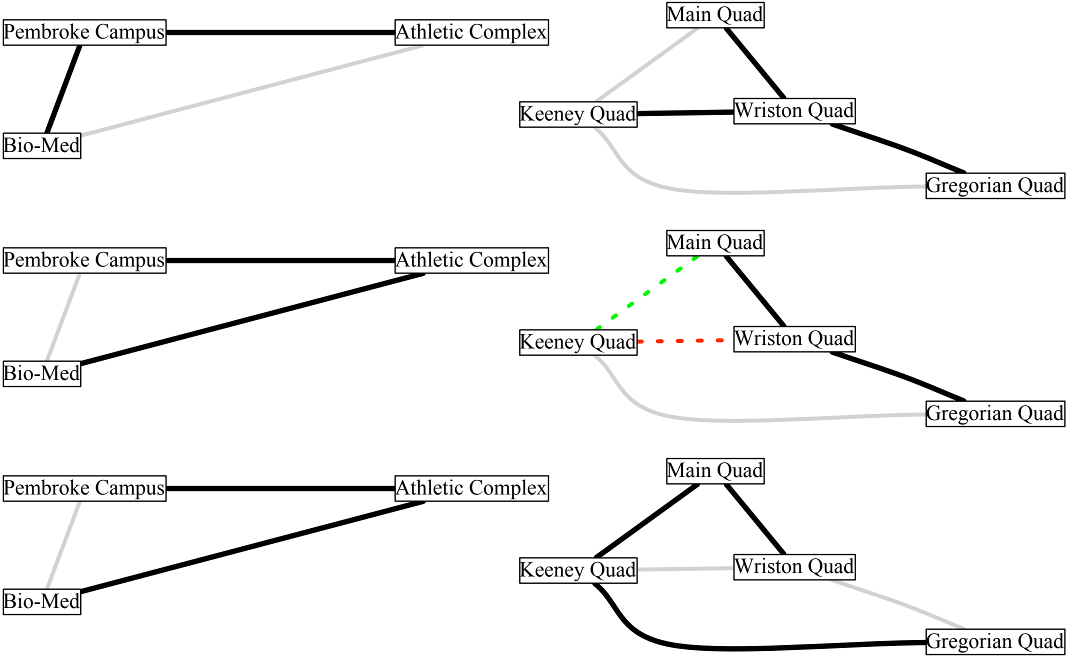

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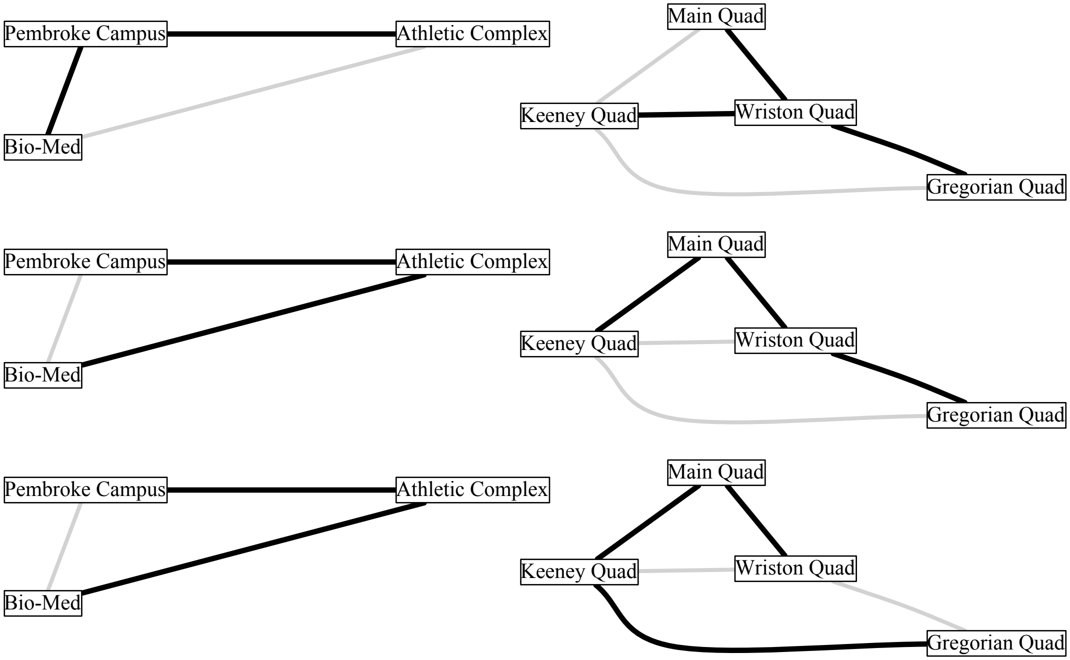

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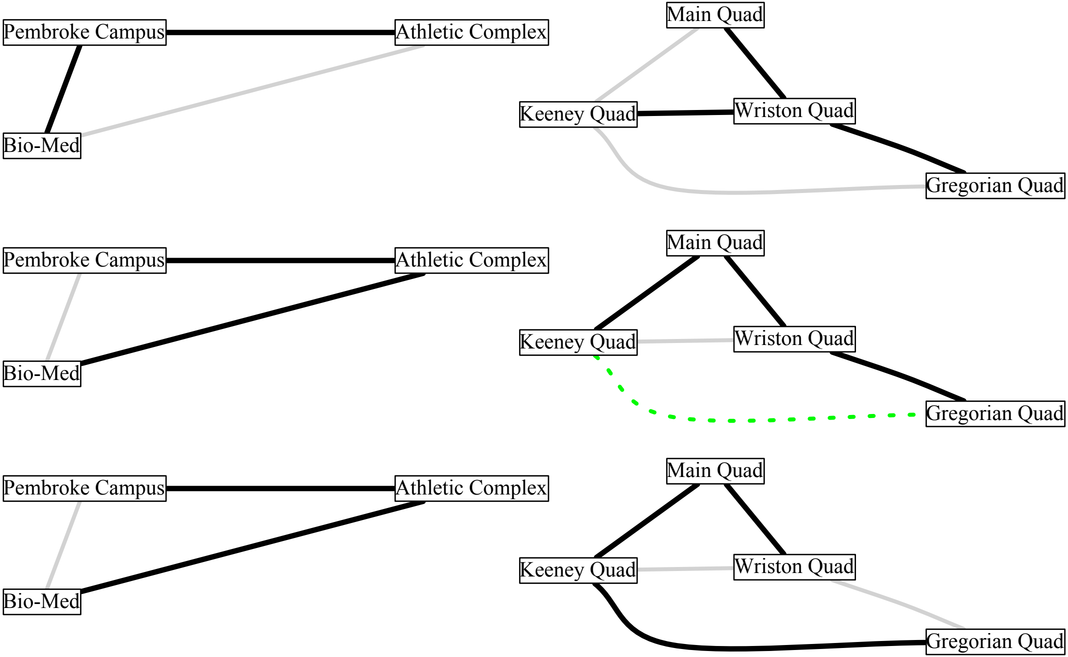

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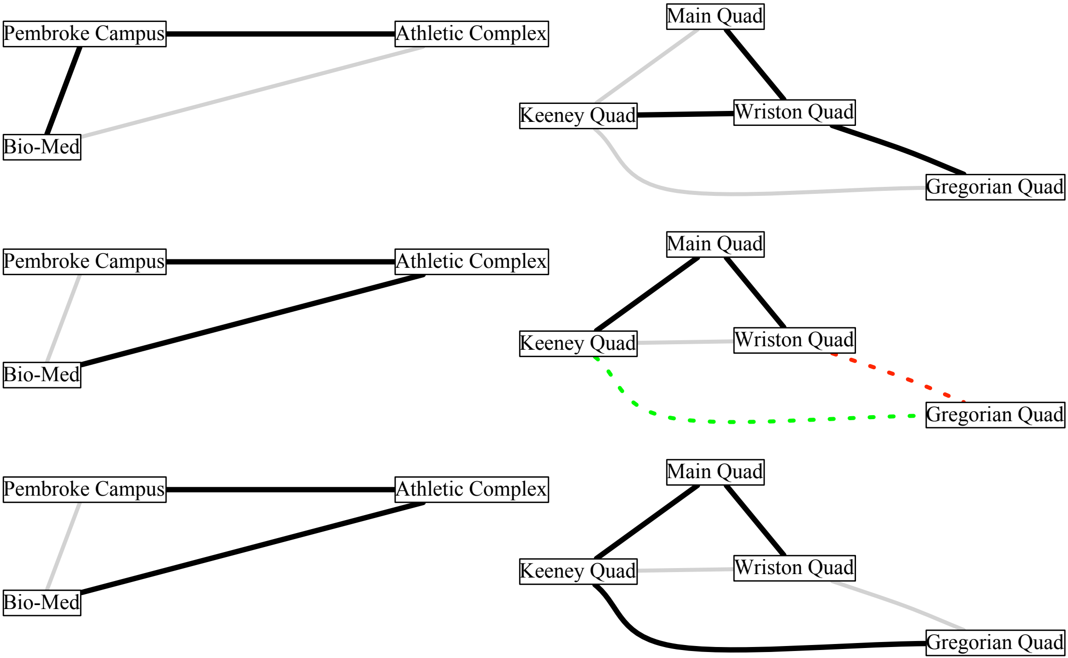

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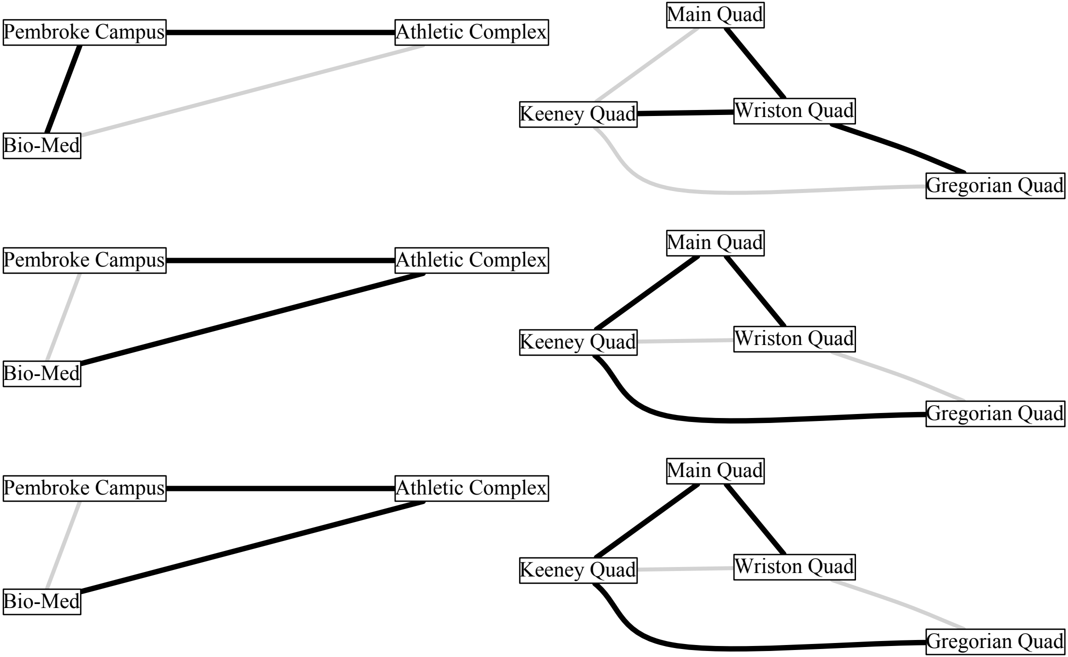

Dimension

Definition: We define the *dimension* of a vector space to be the size of a basis for that vector space. The dimension of a vector space $\mathcal{V}$ is written $\dim \mathcal{V}$.

Definition: We define the *rank* of a set S of vectors as the dimension of Span S. We write rank S.

Example: The vectors $[1, 0, 0], [0, 2, 0], [2, 4, 0]$ are linearly dependent.
Therefore their rank is less than three.
First two of these vectors form a basis for the span of all three, so the rank is two.

Example: The vector space Span $\{[0, 0, 0]\}$ is spanned by an empty set of vectors.
Therefore the rank of $\{[0, 0, 0]\}$ is zero.

Row rank, column rank

Definition: For a matrix M, the *row rank* of M is the rank of its rows, and the *column rank* of M is the rank of its columns.
Equivalently, the row rank of M is the dimension of Row M, and the column rank of M is the dimension of Col M.

Example: Consider the matrix

$$M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 2 & 4 & 0 \end{bmatrix}$$

whose rows are the vectors we saw before: $[1, 0, 0], [0, 2, 0], [2, 4, 0]$
The set of these vectors has rank two, so the row rank of M is two.
The columns of M are $[1, 0, 2]$, $[0, 2, 4]$, and $[0, 0, 0]$.
Since the third vector is the zero vector, it is not needed for spanning the column space.
Since each of the first two vectors has a nonzero where the other has a zero, these two are linearly independent, so the column rank is two.

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Example: Consider the matrix

$$M = \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 2 & 0 & 7 \\ 0 & 0 & 3 & 9 \end{bmatrix}$$

Each of the rows has a nonzero where the others have zeroes, so the three rows are linearly independent. Thus the row rank of M is three.
The columns of M are $[1, 0, 0]$, $[0, 2, 0]$, $[0, 0, 3]$, and $[5, 7, 9]$.
The first three columns are linearly independent, and the fourth can be written as a linear combination of the first three, so the column rank is three.

Row rank, column rank

Definition: For a matrix M, the *row rank* of M is the rank of its rows, and the *column rank* of M is the rank of its columns.
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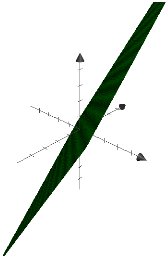
Does column rank always equal row rank? ☺

Geometry

We have asked:

Fundamental Question: How can we predict the dimensionality of the span of some vectors?



Now we can answer:

Compute the rank of the set of vectors.

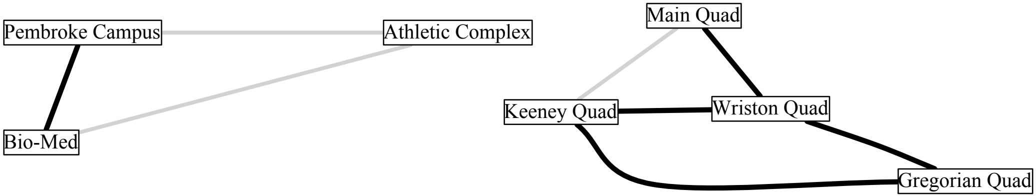
Examples:

• Span $\{[1, 2, -2]\}$ is a line but Span $\{[0, 0, 0]\}$ is a point.
First vector space has dimension one, second has dimension zero.

• Span $\{[1, 2], [3, 4]\}$ consists of all of $\mathbb{R}^2$ but Span $\{[1, 3], [2, 6]\}$ is a line
The first has dimension two and the second has dimension one.

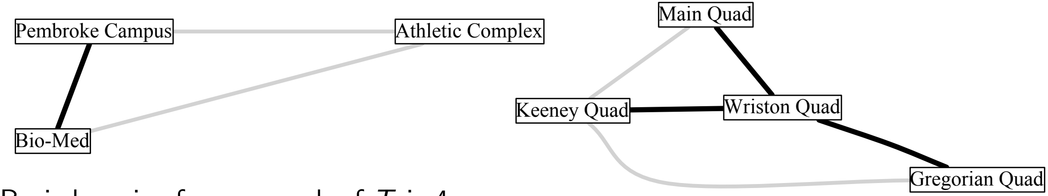
• Span $\{[1, 0, 0], [0, 1, 0], [0, 0, 1]\}$ is $\mathbb{R}^3$ but Span $\{[1, 0, 0], [0, 1, 0], [1, 1, 0]\}$ is a plane.
The first has dimension three and the second has dimension two.

Dimension and rank in graphs



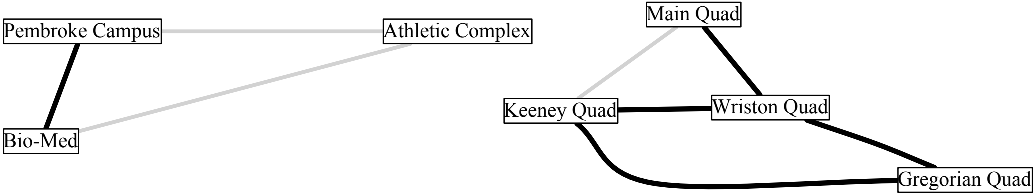
Let T = set of dark edges

Basis for Span T:



Basis has size four, so rank of T is 4.

Dimension and rank in graphs



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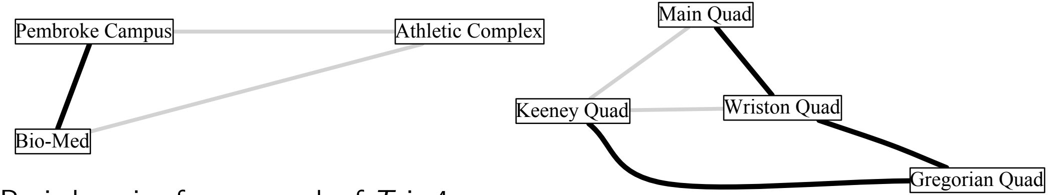

Basis has size four, so rank of T is 4.

Cardinality of a vector space over $GF(2)$

Recall *checksum problem*
Checksum function $\mathbf{x} \mapsto [\mathbf{a}_1 \cdot \mathbf{x}, \ldots, \mathbf{a}_{64} \cdot \mathbf{x}]$

Original "file" $\mathbf{p}$, transmission error $\mathbf{e}$ so corrupted file is $\mathbf{p} + \mathbf{e}$.

What is probability that corrupted file has the same checksum as original?

If error is chosen according to uniform distribution,

$$\begin{aligned} &\text{Probability}\,(\mathbf{p} + \mathbf{e} \text{ has same checksum as } \mathbf{p}) \\ &= \text{Probability}\,(\mathbf{e} \text{ is a solution to homogeneous linear system}) \\ &= \frac{\text{number of solutions to homogeneous linear system}}{\text{number of } n\text{-vectors}} \\ &= \frac{\text{number of solutions to homogeneous linear system}}{2^n} \end{aligned}$$

raising Question

$\boxed{\text{How to find number of solutions to a homogeneous linear system over } GF(2)\text{?}}$

Cardinality of a vector space over $GF(2)$

How to find number of solutions to a homogeneous linear system over $GF(2)$?

Solution set of a homogeneous linear system is a vector space.
Question becomes

How to find out cardinality of a vector space $\mathcal{V}$ over $GF(2)$?

- Suppose basis for $\mathcal{V}$ is $\mathbf{b}_1, \ldots, \mathbf{b}_n$.
- Then $\mathcal{V}$ is set of linear combinations
$$\beta_1\,\mathbf{b}_1 + \cdots + \beta_n\,\mathbf{b}_n$$
- Number of linear combinations is 2^n.
- By Unique-Representation Lemma, every linear combination gives a different vector of $\mathcal{V}$.
- Thus cardinality is $2^{\dim \mathcal{V}}$.

Cardinality of a vector space over $GF(2)$

Cardinality of a vector space $\mathcal{V}$ over $GF(2)$ is $2^{\dim \mathcal{V}}$.

$\boxed{\text{How to find dimension of solution set of a homogeneous linear system?}}$

Write linear system as $A\mathbf{x} = \mathbf{0}$.

$\boxed{\text{How to find dimension of the null space of } A\text{?}}$

Answers will come later.

Subset-Basis Lemma

Lemma: Every finite set T of vectors contains a subset S that is a basis for Span T.

Proof: The Grow algorithm finds a basis for $\mathcal{V}$ if it terminates.

> Initialize $S = \emptyset$.
> Repeat while possible: select a vector $\mathbf{v}$ in $\mathcal{V}$ that is not in Span S, and put it in S.

Revised version:

> Initialize $S = \emptyset$
> Repeat while possible: select a vector $\mathbf{v}$ in T that is not in Span S, and put it in S.

Differs from original:

- This algorithm stops when Span S contains every vector in T.
- The original Grow algorithm stops only once Span S contains every vector in $\mathcal{V}$.

However, that's okay: when Span S contains all the vectors in T, Span S also contains all linear combinations of vectors in T, so at this point Span $S = \mathcal{V}$.
Shows that original Grow algorithm can be guided to make same choices as this algorithm, so result is a basis. QED

Termination of Grow algorithm

def $\textsc{Grow}(\mathcal{V})$
  $B = \emptyset$
  repeat while possible:
    find a vector $\mathbf{v}$ in $\mathcal{V}$ that is not in Span B, and put it in S.

Grow-Algorithm-Termination Lemma: If $\mathcal{V}$ is a subspace of $\mathbb{F}^D$ where D is finite then $\textsc{Grow}(\mathcal{V})$ terminates.
Proof: By Grow-Algorithm Corollary, B is linearly independent throughout.

Apply the Morphing Lemma with $S = \{\text{standard generators for } \mathbb{F}^D\} \Rightarrow$
$|B| \leq |S| = |D|$.

Since B grows in each iteration, there are at most $|D|$ iterations. QED

Every subspace of $\mathbb{F}^D$ contains a basis

Grow-Algorithm-Termination Lemma: If $\mathcal{V}$ is a subspace of $\mathbb{F}^D$ where D is finite then $\textsc{Grow}(\mathcal{V})$ terminates.
Theorem: For finite D, every subspace of $\mathbb{F}^D$ contains a basis.
Proof: Let $\mathcal{V}$ be a subspace of $\mathbb{F}^D$.

```
def GROW(V)
  B = ∅
  repeat while possible:
        find a vector v in V that is not in Span B, and put it in B.
```

Grow-Algorithm-Termination Lemma ensures algorithm terminates.
Upon termination, every vector in $\mathcal{V}$ is in Span B, so B is a set of generators for $\mathcal{V}$. By Grow-Algorithm Corollary, B is linearly independent. Therefore B is a basis for $\mathcal{V}$.
QED

Superset-Basis Lemma

Grow-Algorithm-Termination Lemma: If $\mathcal{V}$ is a subspace of $\mathbb{F}^D$ where D is finite then $\textsc{Grow}(\mathcal{V})$ terminates.

Superset-Basis Lemma: Let $\mathcal{V}$ be a vector space consisting of D-vectors where D is finite. Let C be a linearly independent set of vectors belonging to $\mathcal{V}$. Then $\mathcal{V}$ has a basis B containing all vectors in C.

Proof: Use version of Grow algorithm:

> Initialize B to the empty set.
> Repeat while possible: select a vector $\mathbf{v}$ in $\mathcal{V}$ (preferably in C) that is not in Span B, and put it in B.

At first, B will consist of vectors in C until B contains all of C. Then more vectors will be added to B until Span $B = \mathcal{V}$ By Grow-Algorithm Corollary, B is linearly independent throughout. Therefore, once algorithm terminates, B contains C and is a basis for $\mathcal{U}$.

Termination is implied by Grow Algorithm Termination Lemma. QED

Estimating dimension

$T = \{[-0.6, -2.1, -3.5, -2.2], [-1.3, 1.5, -0.9, -0.5], [4.9, -3.7, 0.5, -0.3],$
$[2.6, -3.5, -1.2, -2.0], [-1.5, -2.5, -3.5, 0.94]\}.$
What is the rank of T?

By Subset-Basis Lemma, T contains a basis.

Therefore $\dim \text{Span } T \leq |T|$.

Therefore rank $T \leq |T|$.

Proposition: A set T of vectors has rank $\leq |T|$.

Dimension Lemma

Dimension Lemma: If $\mathcal{U}$ is a subspace of $\mathcal{W}$ then

- **D1:** $\dim \mathcal{U} \leq \dim \mathcal{W}$, and
- **D2:** if $\dim \mathcal{U} = \dim \mathcal{W}$ then $\mathcal{U} = \mathcal{W}$

Proof: Let $\mathbf{u}_1, \ldots, \mathbf{u}_k$ be a basis for $\mathcal{U}$.
By Superset-Basis Lemma, there is a basis B for $\mathcal{W}$ that contains $\mathbf{u}_1, \ldots, \mathbf{u}_k$.

- $B = \{\mathbf{u}_1, \ldots, \mathbf{u}_k, \mathbf{b}_1, \ldots, \mathbf{b}_r\}$
- Thus $k \leq |B|$, and
- If $k = |B|$ then $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\} = B$ QED

Example: Suppose $\mathcal{V} = \text{Span } \{[1,2],[2,1]\}$.
Clearly $\mathcal{V}$ is a subspace of $\mathbb{R}^2$.
However, the set $\{[1,2],[2,1]\}$ is linearly independent, so $\dim \mathcal{V} = 2$.
Since $\dim \mathbb{R}^2 = 2$, D2 shows that $\mathcal{V} = \mathbb{R}^2$.
Example: $S = \{[-0.6, -2.1, -3.5, -2.2], [-1.3, 1.5, -0.9, -0.5], [4.9, -3.7, 0.5, -0.3],$
$[2.6, -3.5, -1.2, -2.0], [-1.5, -2.5, -3.5, 0.94]\}$
Since every vector in S is a 4-vector, Span S is a subspace of $\mathbb{R}^4$.
Since $\dim \mathbb{R}^4 = 4$, D1 shows $\dim \text{Span } S \leq 4$.

Proposition: Any set of D-vectors has rank at most $|D|$.

Rank Theorem

Rank Theorem: For every matrix M, row rank equals column rank.

Lemma: For any matrix A, row rank of $A \leq$ column rank of A
To show theorem:

- Apply lemma to $M \Rightarrow$ row rank of $M \leq$ column rank of M
- Apply lemma to $M^T \Rightarrow$ row rank of $M^T \leq$ column rank of $M^T \Rightarrow$ column rank of $M \leq$ row rank of M

Combine $\Rightarrow$ row rank of $M =$ column rank of M

Proof of lemma: For any matrix A, row rank of $A \leq$ column rank of A

$$\left[\begin{array}{c} \\ \\ \text{A} \\ \\ \\ \end{array} \right]$$

Think of A as columns $\mathbf{a}_1, \ldots, \mathbf{a}_n$.
Let $\mathbf{b}_1, \ldots, \mathbf{b}_r$ be basis for column space (so column rank $= r$).

Write each column of A in terms of basis: $\left[\begin{array}{c} \\ \mathbf{a}_j \\ \\ \end{array} \right] = \left[\begin{array}{c|c|c} & & \\ \mathbf{b}_1 & \cdots & \mathbf{b}_r \\ & & \end{array} \right] \left[\begin{array}{c} \\ \mathbf{u}_j \\ \\ \end{array} \right]$

Use matrix-vector definition of matrix-matrix multiplication to rewrite as matrix-matrix equation $A = BU$.
B has r columns and U has r rows.
Write A and B in terms of rows: row i of A equals row i of B times U.
Write U in terms of rows: row i of A is a linear combination of rows of U.
Each row of A is in span of the r rows of U. **Thus row rank of A is at most r.**

Proof of lemma: For any matrix A, row rank of $A \leq$ column rank of A

$$\left[\begin{array}{c|c|c|c|c|c|c|c|c} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & a_9 \end{array}\right]$$

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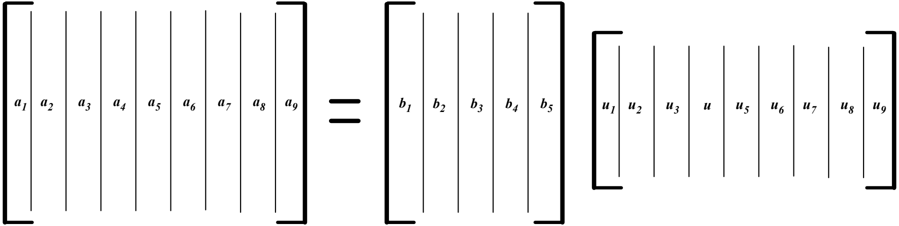

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Proof of lemma: For any matrix A, row rank of $A \leq$ column rank of A

$$\left[\begin{array}{c|c|c|c|c|c|c|c|c} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & a_9 \end{array}\right] = \left[\begin{array}{c|c|c|c|c} b_1 & b_2 & b_3 & b_4 & b_5 \end{array}\right] \left[\begin{array}{ccc} & \mathrm{U} & \end{array}\right]$$

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Proof of lemma: For any matrix A, row rank of $A \leq$ column rank of A

$$\left[\quad A \quad\right] = \left[\quad B \quad\right]\left[\quad U \quad\right]$$

Think of A as columns $\mathbf{a}_1, \ldots, \mathbf{a}_n$.
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Proof of lemma: For any matrix A, row rank of $A \leq$ column rank of A

$$\left[\begin{array}{c} \bar{a}_1 \\ \hline \bar{a}_1 \\ \hline \bar{a}_1 \\ \hline \bar{a}_1 \\ \hline \bar{a}_1 \\ \hline \bar{a}_1 \\ \hline \bar{a}_1 \end{array}\right] = \left[\begin{array}{c} \bar{b}_1 \\ \hline \bar{b}_1 \\ \hline \bar{b}_1 \\ \hline \bar{b}_1 \\ \hline \bar{b}_1 \\ \hline \bar{b}_1 \\ \hline \bar{b}_1 \end{array}\right] \left[\begin{array}{c} \quad \mathrm{U} \quad \end{array}\right]$$

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Each row of A is in span of the r rows of U. **Thus row rank of A is at most r.**

Proof of lemma: For any matrix A, row rank of $A \leq$ column rank of A

$$\begin{bmatrix} \bar{a}_1 \\ \hline \bar{a}_2 \\ \hline \bar{a}_3 \\ \hline \bar{a}_4 \\ \hline \bar{a}_5 \\ \hline \bar{a}_6 \\ \hline \bar{a}_7 \end{bmatrix} = \begin{bmatrix} \bar{b}_1 \\ \hline \bar{b}_2 \\ \hline \bar{b}_3 \\ \hline \bar{b}_4 \\ \hline \bar{b}_5 \\ \hline \bar{b}_6 \\ \hline \bar{b}_7 \end{bmatrix} \begin{bmatrix} \bar{u}_1 \\ \hline \bar{u}_2 \\ \hline \bar{u}_3 \\ \hline \bar{u}_4 \\ \hline \bar{u}_5 \end{bmatrix}$$

Think of A as columns $\mathbf{a}_1, \ldots, \mathbf{a}_n$.
Let $\mathbf{b}_1, \ldots, \mathbf{b}_r$ be basis for column space (so column rank $= r$).

Write each column of A in terms of basis: $\left[\ \mathbf{a}_j\ \right] = \left[\ \mathbf{b}_1 \ \middle|\ \cdots \ \middle|\ \mathbf{b}_r\ \right] \left[\ \mathbf{u}_j\ \right]$

Use matrix-vector definition of matrix-matrix multiplication to rewrite as matrix-matrix equation $A = BU$.
B has r columns and U has r rows.
Write A and B in terms of rows: row i of A equals row i of B times U.
Write U in terms of rows: row i of A is a linear combination of rows of U.
Each row of A is in span of the r rows of U. **Thus row rank of A is at most r.**

Simple authentication revisited

- Password is an n-vector $\hat{\mathbf{x}}$ over $GF(2)$
- **Challenge:** Computer sends random n-vector $\mathbf{a}$
- **Response:** Human sends back $\mathbf{a} \cdot \hat{\mathbf{x}}$.

Repeated until Computer is convinced that Human knows password $\hat{\mathbf{x}}$.

Eve eavesdrops on communication, learns m pairs

$$\begin{array}{c} \mathbf{a}_1, b_1 \\ \vdots \\ \mathbf{a}_m, b_m \end{array}$$

such that b_i is right response to challenge $\mathbf{a}_i$

Then Eve can calculate right response to any challenge in Span $\{\mathbf{a}_1, \ldots, \mathbf{a}_m\}$:

Suppose $\mathbf{a} = \alpha_1 \mathbf{a}_1 + \cdots + \alpha_m \mathbf{a}_m$
Then right response is $\alpha_1 b_1 + \cdots + \alpha_m b_m$

Fact: Probably rank $[\mathbf{a}_1, \ldots, \mathbf{a}_m]$ is not much less than $\min\{m, n\}$.

Once $m > n$, probably Span $\{\mathbf{a}_1, \ldots, \mathbf{a}_m\}$ is all of $GF(2)^n$
so Eve can respond to any challenge.

Also: The password $\hat{\mathbf{x}}$ is a solution to

$$\underbrace{\begin{bmatrix} \mathbf{a}_1 \\ \hline \vdots \\ \hline \mathbf{a}_m \end{bmatrix}}_{A} \begin{bmatrix} \mathbf{x} \end{bmatrix} = \underbrace{\begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}}_{\mathbf{b}}$$

Solution set of $A\mathbf{x} = \mathbf{b}$ is $\hat{\mathbf{x}} + \text{Null } A$

Once rank A reaches n, columns of A are linearly independent so Null A is trivial, so only solution is the password $\hat{\mathbf{x}}$, so Eve can compute the password using `solver`.

Direct Sum

Let $\mathcal{U}$ and $\mathcal{V}$ be two vector spaces consisting of D-vectors over a field $\mathbb{F}$.

Definition: If $\mathcal{U}$ and $\mathcal{V}$ share only the zero vector then we define the *direct sum* of $\mathcal{U}$ and $\mathcal{V}$ to be the set

$$\{\mathbf{u} + \mathbf{v} \ : \mathbf{u} \in \mathcal{U}, \mathbf{v} \in \mathcal{V}\}$$

written $\mathcal{U} \oplus \mathcal{V}$

That is, $\mathcal{U} \oplus \mathcal{V}$ is the set of all sums of a vector in $\mathcal{U}$ and a vector in $\mathcal{V}$.

In Python, `[u+v for u in U for v in V]`

(But generally $\mathcal{U}$ and $\mathcal{V}$ are infinite so the Python is just suggestive.)

Direct Sum: Example

Vectors over $GF(2)$:

Example: Let $\mathcal{U} = \text{Span } \{1000, 0100\}$ and let $\mathcal{V} = \text{Span } \{0010\}$.

- Every nonzero vector in $\mathcal{U}$ has a one in the first or second position (or both) and nowhere else.
- Every nonzero vector in $\mathcal{V}$ has a one in the third position and nowhere else.

Therefore the only vector in both $\mathcal{U}$ and $\mathcal{V}$ is the zero vector.

Therefore $\mathcal{U} \oplus \mathcal{V}$ is defined.

$\mathcal{U} \oplus \mathcal{V} = \{0000 + 0000, 1000 + 0000, 0100 + 0000, 1100 + 0000, 0000 + 0010, 1000 + 0010, 0100 + 0010, 1100 + 0010\}$

which is equal to $\{0000, 1000, 0100, 1100, 0010, 1010, 0110, 1110\}$.

Direct Sum: Example

Vectors over $\mathbb{R}$:

Example: Let $\mathcal{U} = \text{Span } \{[1,2,1,2],[3,0,0,4]\}$ and let $\mathcal{V}$ be the null space of $\begin{bmatrix} 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix}$.

- The vector $[2,-2,-1,2]$ is in $\mathcal{U}$ because it is $[3,0,0,4]-[1,2,1,2]$
- It is also in $\mathcal{V}$ because

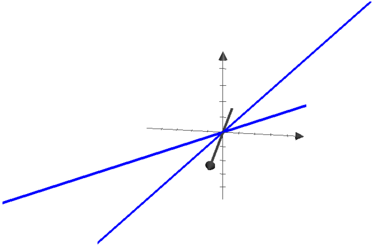
$$\begin{bmatrix} 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ -2 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Therefore we cannot form $\mathcal{U} \oplus \mathcal{V}$.

Direct Sum: Example

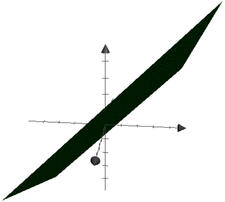
Vectors over $\mathbb{R}$:

Example:



- Let $\mathcal{U} = \text{Span } \{[4, -1, 1]\}$.
- Let $\mathcal{V} = \text{Span } \{[0, 1, 1]\}$.

The only intersection is at the origin, so $\mathcal{U} \oplus \mathcal{V}$ is defined.



- $\mathcal{U} \oplus \mathcal{V}$ is the set of vectors $\mathbf{u} + \mathbf{v}$ where $\mathbf{u} \in \mathcal{U}$ and $\mathbf{v} \in \mathcal{V}$.
- This is just $\text{Span } \{[4, -1, 1], [0, 1, 1]\}$
- Plane containing the two lines

Properties of direct sum

Lemma: $\mathcal{U} \oplus \mathcal{V}$ is a vector space.

(Prove using Properties V1, V2, V3.)

Lemma: The union of

- a set of generators of $\mathcal{U}$, and
- a set of generators of $\mathcal{V}$

is a set of generators for $\mathcal{U} \oplus \mathcal{V}$.

Proof: Suppose $\mathcal{U} = \text{Span } \{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$ and $\mathcal{V} = \text{Span } \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$.
Then

- every vector in $\mathcal{U}$ can be written as $\alpha_1 \mathbf{u}_1 + \cdots + \alpha_m \mathbf{u}_m$, and
- every vector in $\mathcal{V}$ can be written as $\beta_1 \mathbf{v}_1 + \cdots + \beta_n \mathbf{v}_n$

so every vector in $\mathcal{U} \oplus \mathcal{V}$ can be written as

$$\alpha_1 \mathbf{u}_1 + \cdots + \alpha_m \mathbf{u}_m \; + \; \beta_1 \mathbf{v}_1 + \cdots + \beta_n \mathbf{v}_n$$

QED

Properties of direct sum

Direct Sum Basis Lemma:

Union of a basis of $\mathcal{U}$ and a basis of $\mathcal{V}$ is a basis of $\mathcal{U} \oplus \mathcal{V}$.

Proof: Clearly

- a basis of $\mathcal{U}$ is a set of generators for $\mathcal{U}$, and
- a basis of $\mathcal{V}$ is a set of generators for $\mathcal{V}$.

Therefore the previous lemma shows that

- the union of a basis for $\mathcal{U}$ and a basis for $\mathcal{V}$ is a generating set for $\mathcal{U} \oplus \mathcal{V}$.

We just need to show that the union is linearly independent.

Properties of direct sum

Direct Sum Basis Lemma:

Union of a basis of $\mathcal{U}$ and a basis of $\mathcal{V}$ is a basis of $\mathcal{U} \oplus \mathcal{V}$.

Proof, cont'd: Let $\{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$ be a basis for $\mathcal{U}$. Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a basis for $\mathcal{V}$.
We need to show that $\{\mathbf{u}_1, \ldots, \mathbf{u}_m, \mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is independent.
Suppose

$$\mathbf{0} = \alpha_1 \ \mathbf{u}_1 + \cdots + \alpha_m \mathbf{u}_m + \beta_1 \ \mathbf{v}_1 + \cdots + \beta_n \ \mathbf{v}_n.$$

Then

$$\underbrace{\alpha_1 \ \mathbf{u}_1 + \cdots + \alpha_m \ \mathbf{u}_m}_{\text{in } \mathcal{U}} = \underbrace{(-\beta_1) \ \mathbf{v}_1 + \cdots + (-\beta_n) \ \mathbf{v}_n}_{\text{in } \mathcal{V}}$$

Left-hand side is a vector in $\mathcal{U}$, and right-hand side is a vector in $\mathcal{V}$.

By definition of $\mathcal{U} \oplus \mathcal{V}$, the only vector in both $\mathcal{U}$ and $\mathcal{V}$ is the zero vector.

This shows:

$$\mathbf{0} = \alpha_1 \ \mathbf{u}_1 + \cdots + \alpha_m \ \mathbf{u}_m$$

and

$$\mathbf{0} = (-\beta_1) \ \mathbf{v}_1 + \cdots + (-\beta_n) \ \mathbf{v}_n$$

By linear independence, the linear combinations must be trivial. QED

Direct Sum

Direct Sum Basis Lemma:

Union of a basis of $\mathcal{U}$ and a basis of $\mathcal{V}$ is a basis of $\mathcal{U} \oplus \mathcal{V}$.

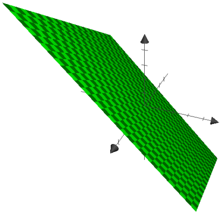
Direct Sum Dimension Corollary: $\dim \mathcal{U} + \dim \mathcal{V} = \dim \mathcal{U} \oplus \mathcal{V}$

Proof: A basis for $\mathcal{U}$ together with a basis for $\mathcal{V}$ forms a basis for $\mathcal{U} \oplus \mathcal{V}$. QED

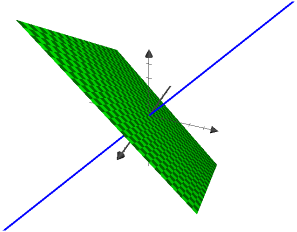
Complementary subspace

If $\mathcal{U} \oplus \mathcal{V} = \mathcal{W}$, we say $\mathcal{U}$ and $\mathcal{V}$ are *complementary* subspaces of $\mathcal{W}$.

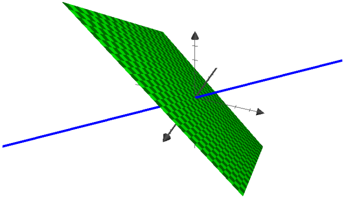
Example: Suppose $\mathcal{U}$ is a plane in $\mathbb{R}^3$.



Then any line through the origin that does not lie in $\mathcal{U}$ is complementary subspace with respect to $\mathbb{R}^3$



Example illustrates that, for a given subspace $\mathcal{U}$ of $\mathcal{W}$, there can be many different subspaces $\mathcal{V}$ such that $\mathcal{U}$ and $\mathcal{V}$ are complementary.



Complementary subspace

Proposition: For any finite-dimensional vector space $\mathcal{W}$ and any subspace $\mathcal{U}$, there is a subspace $\mathcal{V}$ such that $\mathcal{U}$ and $\mathcal{V}$ are complementary.

Proof: Let $\mathbf{u}_1, \ldots, \mathbf{u}_k$ be a basis for $\mathcal{U}$. By Superset-Basis Lemma, there is a basis for $\mathcal{W}$ that includes $\mathbf{u}_1, \ldots, \mathbf{u}_k$:

$$B = \{\mathbf{u}_1, \ldots, \mathbf{u}_k, \mathbf{v}_1, \ldots, \mathbf{v}_r\}$$

Let $\mathcal{V} = \text{Span } \{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$.

Any vector in $\mathcal{W}$ can be written in terms of its basis:

$$\mathbf{w} = \underbrace{\alpha_1\,\mathbf{u}_1 + \cdots + \alpha_k\,\mathbf{u}_k}_{\text{in } \mathcal{U}} + \underbrace{\beta_1\,\mathbf{v}_1 + \cdots + \beta_r\,\mathbf{v}_r}_{\text{in } \mathcal{V}}$$

If some vector $\mathbf{v}$ is in Span $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ and in Span $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$
then $\mathbf{v} = \alpha_1\,\mathbf{u}_1 + \cdots + \alpha_k\,\mathbf{u}_k$ and $\mathbf{v} = \beta_1\,\mathbf{v}_1 + \cdots + \beta_r\,\mathbf{v}_r$
so

$$\alpha_1\,\mathbf{u}_1 + \cdots + \alpha_k\,\mathbf{u}_k = \beta_1\,\mathbf{v}_1 + \cdots + \beta_r\,\mathbf{v}_r$$

$$\mathbf{0} = \alpha_1\,\mathbf{u}_1 + \cdots + \alpha_k\,\mathbf{u}_k - \beta_1\,\mathbf{v}_1 - \cdots - \beta_r\,\mathbf{v}_r$$

so $\alpha_1 = \cdots = \alpha_k = \beta_1 = \cdots = \beta_r = 0$ so $\mathbf{v} = \mathbf{0}$. QED

Linear function invertibility

How to tell if a linear function $f : \mathcal{V} \longrightarrow \mathcal{W}$ is invertible?

- *One-to-one?* f is one-to-one if its kernel is trivial. *Equivalent:* if its kernel has dimension zero.
- *Onto?* f is onto if its image equals its co-domain

Recall that the image of a function f with domain $\mathcal{V}$ is $\{f(\mathbf{v}) \ : \mathbf{v} \in \mathcal{V}\}$.

Lemma: The image of f is a subspace of $\mathcal{W}$.

How can we tell if the image of f equals $\mathcal{W}$?

Dimension Lemma: If $\mathcal{U}$ is a subspace of $\mathcal{W}$ then

Property D1: $\dim \mathcal{U} \leq \dim \mathcal{W}$, and

Property D2: if $\dim \mathcal{U} = \dim \mathcal{W}$ then $\mathcal{U} = \mathcal{W}$

Use Property D2 with $\mathcal{U} = \text{Im } f$.
Shows that the function f is onto iff $\dim \text{Im } f = \dim \mathcal{W}$

We conclude:

$$\boxed{f \text{ is invertible } \dim \text{Ker } f = 0 \text{ and } \dim \text{Im } f = \dim \mathcal{W}}$$

Linear function invertibility

$$\boxed{f \text{ is one-to-one if } \dim \operatorname{Ker} f = 0 \text{ and } \dim \operatorname{Im} f = \dim \mathcal{W}}$$

How does this relate to dimension of the domain?

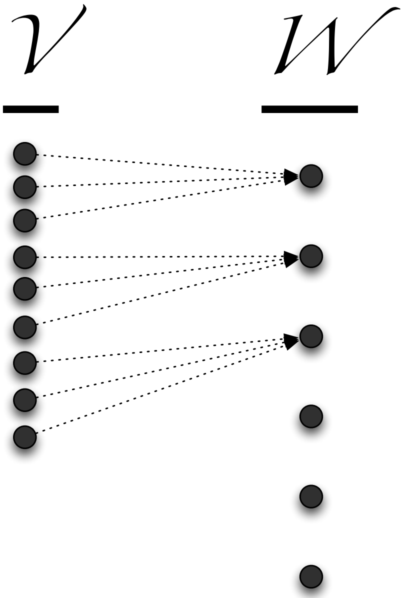
Conjecture: For f to be invertible, need $\dim \mathcal{V} = \dim \mathcal{W}$.

Extracting an invertible function

Starting with a linear function f we will extract a largest possible subfunction that is invertible.

Make it onto by setting co-domain to be image of f.

Make it one-to-one by getting rid of extra domain elements sharing same image.



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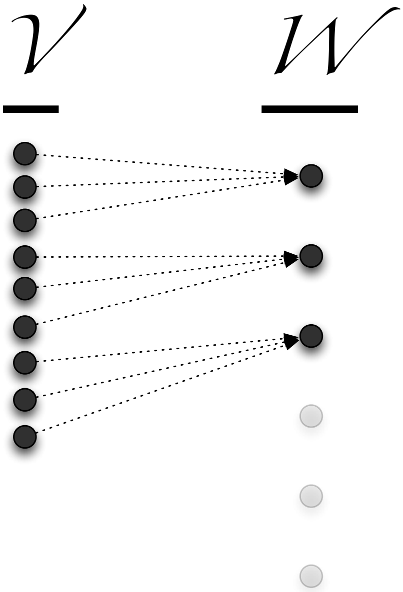


Extracting an invertible function

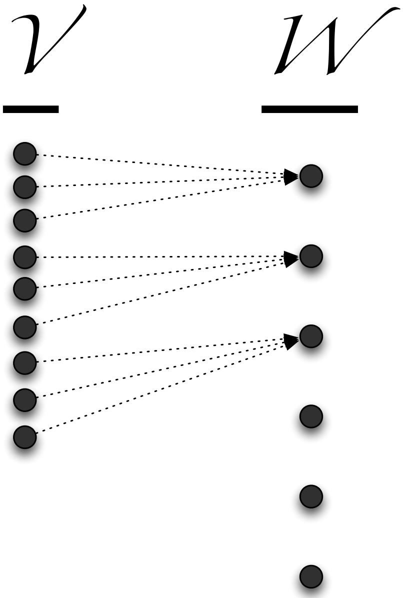
Start with linear function $f : \mathcal{V} \longrightarrow \mathcal{W}$

Step 1: Choose smaller co-domain $\mathcal{W}^*$

Step 2: Choose smaller domain $\mathcal{V}^*$

Step 3: Define function $f^* : \mathcal{V}^* \longrightarrow \mathcal{W}^*$ by $f^*(\mathbf{x}) = f(\mathbf{x})$

In fact, we will end up selecting a *basis* of $\mathcal{W}^*$ and a basis of $\mathcal{V}^*$.



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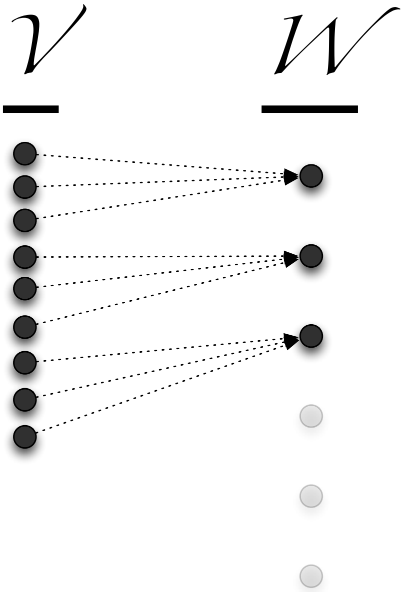

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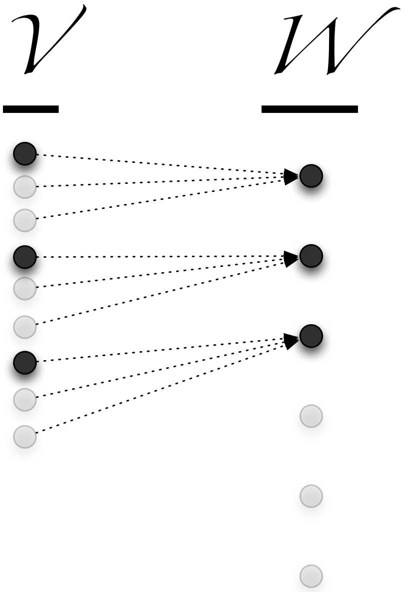
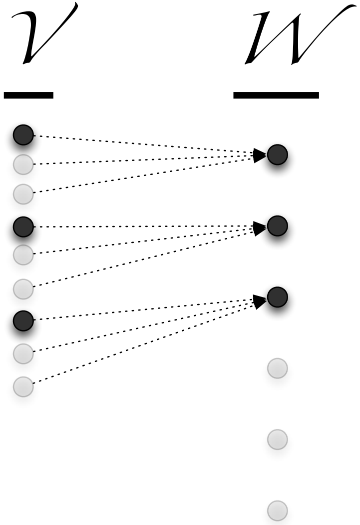


Extracting an invertible function from linear function $f : \mathcal{V} \longrightarrow \mathcal{W}$

- Choose smaller co-domain $\mathcal{W}^*$
  Let $\mathcal{W}^*$ be image of f

  Let $\mathbf{w}_1, \ldots, \mathbf{w}_r$ be a basis of $\mathcal{W}^*$
- Choose smaller domain $\mathcal{V}^*$
  Let $\mathbf{v}_1, \ldots, \mathbf{v}_r$ be pre-images of $\mathbf{w}_1, \ldots, \mathbf{w}_r$
  That is, $f(\mathbf{v}_1) = \mathbf{w}_1, \ldots, f(\mathbf{v}_r) = \mathbf{w}_r$
  Let $\mathcal{V}^* = \text{Span }\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$
- Define function $f^* : \mathcal{V}^* \longrightarrow \mathcal{W}^*$ by $f^*(\mathbf{x}) = f(\mathbf{x})$



We will show:

- f^* is onto
- f^* is one-to-one (kernel is trivial)
- Bonus: $\mathbf{v}_1, \ldots, \mathbf{v}_r$ form a basis for $\mathcal{V}^*$

Extracting an invertible function from linear function $f : \mathcal{V} \longrightarrow \mathcal{W}$

- Choose smaller co-domain $\mathcal{W}^*$
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  by $f^*(\mathbf{x}) = f(\mathbf{x})$

We will show:

- f^* is onto
- f^* is one-to-one (kernel is trivial)
- Bonus: $\mathbf{v}_1, \ldots, \mathbf{v}_r$ form a basis for $\mathcal{V}^*$

Onto:

Let $\mathbf{w}$ be any vector in co-domain $\mathcal{W}^*$.
There are scalars $\alpha_1, \ldots, \alpha_r$ such that

$$\mathbf{w} = \alpha_1 \, \mathbf{w}_1 + \cdots + \alpha_r \, \mathbf{w}_r$$

Because f is linear,

$$\begin{aligned} f(\alpha_1 \, \mathbf{v}_1 + \cdots + \alpha_r \, \mathbf{v}_r) & \\ = \; & \alpha_1 \, f(\mathbf{v}_1) + \cdots + \alpha_r \, f(v_r) \\ = \; & \alpha_1 \, \mathbf{w}_1 + \cdots + \alpha_r \, \mathbf{w}_r \end{aligned}$$

so $\mathbf{w}$ is image of $\alpha_1 \, \mathbf{v}_1 + \cdots + \alpha_r \, \mathbf{v}_r \in \mathcal{V}^*$
QED

Extracting an invertible function from linear function $f : \mathcal{V} \longrightarrow \mathcal{W}$

- Choose smaller co-domain $\mathcal{W}^*$
  Let $\mathcal{W}^*$ be image of f

  Let $\mathbf{w}_1, \ldots, \mathbf{w}_r$ be a basis of $\mathcal{W}^*$
- Choose smaller domain $\mathcal{V}^*$
  Let $\mathbf{v}_1, \ldots, \mathbf{v}_r$ be pre-images of $\mathbf{w}_1, \ldots, \mathbf{w}_r$
  That is, $f(\mathbf{v}_1) = \mathbf{w}_1, \ldots, f(\mathbf{v}_r) = \mathbf{w}_r$
  Let $\mathcal{V}^* = \text{Span } \{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$
- Define function $f^* : \mathcal{V}^* \longrightarrow \mathcal{W}^*$ by $f^*(\mathbf{x}) = f(\mathbf{x})$

We will show:

- f^* is onto
- f^* is one-to-one (kernel is trivial)
- Bonus: $\mathbf{v}_1, \ldots, \mathbf{v}_r$ form a basis for $\mathcal{V}^*$

One-to-one:
By One-to-One Lemma, need only show kernel is trivial.

Suppose $\mathbf{v}^*$ is in $\mathcal{V}^*$ and $f(\mathbf{v}^*) = \mathbf{0}$

Because $\mathcal{V}^* = \text{Span } \{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$, there are scalars $\alpha_1, \ldots, \alpha_r$ such that

$$\mathbf{v}^* = \alpha_1 \mathbf{v}_1 + \cdots + \alpha_r \mathbf{v}_r$$

Applying f to both sides,

$$\begin{aligned} \mathbf{0} &= f(\alpha_1 \mathbf{v}_1 + \cdots + \alpha_r \mathbf{v}_r) \\ &= \alpha_1 \mathbf{w}_1 + \cdots + \alpha_r \mathbf{w}_r \end{aligned}$$

Because $\mathbf{w}_1, \ldots, \mathbf{w}_r$ are linearly independent, $\alpha_1 = \cdots = \alpha_r = 0$

so $\mathbf{v}^* = \mathbf{0}$ QED

Extracting an invertible function from linear function $f : \mathcal{V} \longrightarrow \mathcal{W}$

- Choose smaller co-domain $\mathcal{W}^*$
  Let $\mathcal{W}^*$ be image of f

  Let $\mathbf{w}_1, \ldots, \mathbf{w}_r$ be a basis of $\mathcal{W}^*$
- Choose smaller domain $\mathcal{V}^*$
  Let $\mathbf{v}_1, \ldots, \mathbf{v}_r$ be pre-images of $\mathbf{w}_1, \ldots, \mathbf{w}_r$
  That is, $f(\mathbf{v}_1) = \mathbf{w}_1, \ldots, f(\mathbf{v}_r) = \mathbf{w}_r$
  Let $\mathcal{V}^* = \text{Span } \{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$
- Define function $f^* : \mathcal{V}^* \longrightarrow \mathcal{W}^*$ by $f^*(\mathbf{x}) = f(\mathbf{x})$

We will show:

- f^* is onto
- f^* is one-to-one (kernel is trivial)
- Bonus: $\mathbf{v}_1, \ldots, \mathbf{v}_r$ form a basis for $\mathcal{V}^*$

Bonus: $\mathbf{v}_1, \ldots, \mathbf{v}_r$ form a basis for $\mathcal{V}^*$

Need only show linear independence
Suppose $\mathbf{0} = \alpha_1 \mathbf{v}_1 + \cdots + \alpha_r \mathbf{v}_r$

Applying f to both sides,

$$\begin{aligned} \mathbf{0} &= f(\alpha_1 \mathbf{v}_1 + \cdots + \alpha_r \mathbf{v}_r) \\ &= \alpha_1 \mathbf{w}_1 + \cdots + \alpha_r \mathbf{w}_r \end{aligned}$$

Because $\mathbf{w}_1, \ldots, \mathbf{w}_r$ are linearly independent, $\alpha_1 = \cdots = \alpha_r = 0$. QED

Extracting an invertible function from linear function $f : \mathcal{V} \longrightarrow \mathcal{W}$

- Choose smaller co-domain $\mathcal{W}^*$
  Let $\mathcal{W}^*$ be image of f

  Let $\mathbf{w}_1, \ldots, \mathbf{w}_r$ be a basis of $\mathcal{W}^*$
- Choose smaller domain $\mathcal{V}^*$
  Let $\mathbf{v}_1, \ldots, \mathbf{v}_r$ be pre-images of $\mathbf{w}_1, \ldots, \mathbf{w}_r$
  That is, $f(\mathbf{v}_1) = \mathbf{w}_1, \ldots, f(\mathbf{v}_r) = \mathbf{w}_r$
  Let $\mathcal{V}^* = \text{Span } \{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$
- Define function $f^* : \mathcal{V}^* \longrightarrow \mathcal{W}^*$ by $f^*(\mathbf{x}) = f(\mathbf{x})$

We will show:

- f^* is onto
- f^* is one-to-one (kernel is trivial)
- Bonus: $\mathbf{v}_1, \ldots, \mathbf{v}_r$ form a basis for $\mathcal{V}^*$

Example:

Let $A = \left[\begin{array}{c|c|c} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{array}\right]$, and define $\mathbf{f} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ by $f(\mathbf{x}) = A\mathbf{x}$.

Define $\mathcal{W}^* = \text{Im } f = \text{Col } A = \text{Span } \{[1, 2, 1], [2, 1, 2], [1, 1, 1]\}$.

One basis for $\mathcal{W}^*$ is $\mathbf{w}_1 = [0, 1, 0]$, $\mathbf{w}_2 = [1, 0, 1]$

Pre-images for $\mathbf{w}_1$ and $\mathbf{w}_2$:
$\mathbf{v}_1 = [\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}]$ and $\mathbf{v}_2 = [-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}]$,
for then $A\mathbf{v}_1 = \mathbf{w}_1$ and $A\mathbf{v}_2 = \mathbf{w}_2$.

Let $\mathcal{V}^* = \text{Span } \{\mathbf{v}_1, \mathbf{v}_2\}$

Then $f^* : \mathcal{V}^* \longrightarrow \text{Im } f$ is onto and one-to-one.

Extracting an invertible function from linear function $f : \mathcal{V} \longrightarrow \mathcal{W}$

- Choose smaller co-domain $\mathcal{W}^*$
  Let $\mathcal{W}^*$ be image of f

  Let $\mathbf{w}_1, \ldots, \mathbf{w}_r$ be a basis of $\mathcal{W}^*$
- Choose smaller domain $\mathcal{V}^*$
  Let $\mathbf{v}_1, \ldots, \mathbf{v}_r$ be pre-images of $\mathbf{w}_1, \ldots, \mathbf{w}_r$
  That is, $f(\mathbf{v}_1) = \mathbf{w}_1, \ldots, f(\mathbf{v}_r) = \mathbf{w}_r$
  Let $\mathcal{V}^* = \text{Span } \{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$
- Define function $f^* : \mathcal{V}^* \longrightarrow \mathcal{W}^*$
  by $f^*(\mathbf{x}) = f(\mathbf{x})$

We will show:

- f^* is onto
- f^* is one-to-one (kernel is trivial)
- Bonus: $\mathbf{v}_1, \ldots, \mathbf{v}_r$ form a basis for $\mathcal{V}^*$

To show about original function f:
original domain $\mathcal{V} = \text{Ker } f \oplus \mathcal{V}^*$
Must prove two things:

1. Ker f and $\mathcal{V}^*$ share only zero vector
2. every vector in $\mathcal{V}$ is the sum of a vector in Ker f and a vector in $\mathcal{V}^*$

We already showed kernel of f^* is trivial.
This shows only vector of Ker f in $\mathcal{V}^*$ is zero vector. —thing 1 is proved.

Let $\mathbf{v}$ be any vector in $\mathcal{V}$, and let $\mathbf{w} = f(\mathbf{v})$.
Since f^* is onto, its domain $\mathcal{V}^*$ contains a vector $\mathbf{v}^*$ such that $f(\mathbf{v}^*) = \mathbf{w}$
Therefore $f(\mathbf{v}) = f(\mathbf{v}^*)$ so
$f(\mathbf{v}) - f(\mathbf{v}^*) = \mathbf{0}$ so $f(\mathbf{v} - \mathbf{v}^*) = \mathbf{0}$
Thus $\mathbf{u} = \mathbf{v} - \mathbf{v}^*$ is in Ker f
and $\mathbf{v} = \mathbf{u} + \mathbf{v}^*$ —thing 2 is proved.

Extracting an invertible function from linear function $f : \mathcal{V} \longrightarrow \mathcal{W}$

- Choose smaller co-domain $\mathcal{W}^*$
  Let $\mathcal{W}^*$ be image of f

  Let $\mathbf{w}_1, \ldots, \mathbf{w}_r$ be a basis of $\mathcal{W}^*$
- Choose smaller domain $\mathcal{V}^*$
  Let $\mathbf{v}_1, \ldots, \mathbf{v}_r$ be pre-images of $\mathbf{w}_1, \ldots, \mathbf{w}_r$
  That is, $f(\mathbf{v}_1) = \mathbf{w}_1, \ldots, f(\mathbf{v}_r) = \mathbf{w}_r$
  Let $\mathcal{V}^* = \text{Span } \{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$
- Define function $f^* : \mathcal{V}^* \longrightarrow \mathcal{W}^*$
  by $f^*(\mathbf{x}) = f(\mathbf{x})$

We will show:

- f^* is onto
- f^* is one-to-one (kernel is trivial)
- Bonus: $\mathbf{v}_1, \ldots, \mathbf{v}_r$ form a basis for $\mathcal{V}^*$

original domain $\mathcal{V}$ = Ker $f \oplus \mathcal{V}^*$

Example: Let $A = \left[\begin{array}{c|c|c} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{array}\right]$, and

define $\mathbf{f} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ by $f(\mathbf{x}) = A\mathbf{x}$.

$\mathbf{v}_1 = [\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}]$ and $\mathbf{v}_2 = [-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}]$

$\mathcal{V}^* = \text{Span } \{\mathbf{v}_1, \mathbf{v}_2\}$

Ker $f = \text{Span } \{[1, 1, -3]\}$

Therefore
$\mathcal{V} = (\text{Span } \{[1, 1, -3]\}) \oplus (\text{Span } \{\mathbf{v}_1, \mathbf{v}_2\})$

Extracting an invertible function from linear function $f : \mathcal{V} \longrightarrow \mathcal{W}$

- Choose smaller co-domain $\mathcal{W}^*$
  Let $\mathcal{W}^*$ be image of f

  Let $\mathbf{w}_1, \ldots, \mathbf{w}_r$ be a basis of $\mathcal{W}^*$

- Choose smaller domain $\mathcal{V}^*$
  Let $\mathbf{v}_1, \ldots, \mathbf{v}_r$ be pre-images of $\mathbf{w}_1, \ldots, \mathbf{w}_r$
  That is, $f(\mathbf{v}_1) = \mathbf{w}_1, \ldots, f(\mathbf{v}_r) = \mathbf{w}_r$
  Let $\mathcal{V}^* = \text{Span } \{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$

- Define function $f^* : \mathcal{V}^* \longrightarrow \mathcal{W}^*$
  by $f^*(\mathbf{x}) = f(\mathbf{x})$

We will show:

- f^* is onto
- f^* is one-to-one (kernel is trivial)
- Bonus: $\mathbf{v}_1, \ldots, \mathbf{v}_r$ form a basis for $\mathcal{V}^*$

original domain $\mathcal{V} = \textbf{Ker } f \oplus \mathcal{V}^*$

By Direct-Sum Dimension Corollary,

$$\dim \mathcal{V} = \dim \text{Ker } f + \dim \mathcal{V}^*$$

Since $\mathbf{v}_1, \ldots, \mathbf{v}_r$ form a basis for $\mathcal{V}^*$,

$$\dim \mathcal{V}^* = r = \dim \text{Im } f$$

We have proved...

Kernel-Image Theorem:
For any linear function $f : \mathcal{V} \rightarrow W$,

$$\dim \text{Ker } f + \dim \text{Im } f = \dim \mathcal{V}$$

Linear function invertibility, revisited

Kernel-Image Theorem:
For any linear function $f : \mathcal{V} \to W$,

$$\dim \text{Ker } f + \dim \text{Im } f = \dim \mathcal{V}$$

Linear-Function Invertibility Theorem: Let $f : \mathcal{V} \longrightarrow \mathcal{W}$ be a linear function. Then f is invertible iff $\dim \text{Ker } f = 0$ and $\dim \mathcal{V} = \dim \mathcal{W}$.

Proof: We saw before that f

- is one-to-one iff $\dim \text{Ker } f = 0$
- is onto if $\dim \text{Im } f = \dim \mathcal{W}$

Therefore f is invertible if $\dim \text{Ker } f = 0$ and $\dim \text{Im } f = \dim \mathcal{W}$.

Kernel-Image Theorem states $\dim \text{Ker } f + \dim \text{Im } f = \dim \mathcal{V}$

Therefore

$$\begin{gathered} \dim \text{Ker } f = 0 \text{ and } \dim \text{Im } f = \dim \mathcal{W} \\ \text{iff} \\ \dim \text{Ker } f = 0 \text{ and } \dim \mathcal{V} = \dim \mathcal{W} \end{gathered}$$

QED

Rank-Nullity Theorem

Kernel-Image Theorem:
For any linear function $f : \mathcal{V} \to W$,

$$\dim \text{Ker } f + \dim \text{Im } f = \dim \mathcal{V}$$

Apply Kernel-Image Theorem to the function $f(\mathbf{x}) = A\mathbf{x}$:

- $\text{Ker } f = \text{Null } A$
- $\dim \text{Im } f = \dim \text{Col } A = \text{rank } A$

Definition: The *nullity* of matrix A is $\dim \text{Null } A$

Rank-Nullity Theorem: For any n-column matrix A,

$$\text{nullity } A + \text{rank } A = n$$

Checksum problem revisited

Checksum function maps n-vectors over $GF(2)$ to 64-vectors over $GF(2)$:

$$\mathbf{x} \mapsto [\mathbf{a}_1 \cdot \mathbf{x}, \ldots, \mathbf{a}_{64} \cdot \mathbf{x}]$$

Original "file" $\mathbf{p}$, transmission error $\mathbf{e}$
so corrupted file is $\mathbf{p} + \mathbf{e}$.

If error is chosen according to uniform distribution,
Probability ($\mathbf{p} + \mathbf{e}$ has same checksum as $\mathbf{p}$)

$$= \frac{2^{\dim \mathcal{V}}}{2^n}$$

where $\mathcal{V}$ is the null space of the matrix

$$A = \begin{bmatrix} \mathbf{a}_1 \\ \hline \vdots \\ \hline \mathbf{a}_{64} \end{bmatrix}$$

Fact: Can easily choose $\mathbf{a}_1, \ldots, \mathbf{a}_{64}$ so that
rank $A = 64$

(Randomly chosen vectors will probably work.)

Rank-Nullity Theorem $\Rightarrow$

$$\begin{array}{ccccc} \text{rank } A & + & \text{nullity } A & = & n \\ 64 & + & \dim \mathcal{V} & = & n \\ & & \dim \mathcal{V} & = & n - 64 \end{array}$$

Therefore
Probability $= \frac{2^{n-64}}{2^n} = \frac{1}{2^{64}}$

very tiny chance that the change is undetected

Matrix invertibility

Rank-Nullity Theorem: For any n-column matrix A,

$$\text{nullity } A + \text{rank } A = n$$

Corollary: Let A be an $R \times C$ matrix. Then A is invertible if and only if $|R| = |C|$ and the columns of A are linearly independent.

Proof: Let $\mathbb{F}$ be the field. Define $f : \mathbb{F}^C \longrightarrow \mathbb{F}^R$ by $f(\mathbf{x}) = A\mathbf{x}$.
Then A is an invertible matrix if and only if f is an invertible function.

$$\begin{array}{lll} \text{The function } f \text{ is invertible} & \text{iff} & \dim \operatorname{Ker} f = 0 \text{ and } \dim \mathbb{F}^C = \dim \mathbb{F}^R \\ & \text{iff} & \text{nullity } A = 0 \text{ and } |C| = |R|. \end{array}$$

$$\begin{array}{lll} \text{nullity } A = 0 & \text{iff} & \dim \operatorname{Null} A = 0 \\ & \text{iff} & \operatorname{Null} A = \{\mathbf{0}\} \\ & \text{iff} & \text{the only vector } \mathbf{x} \text{ such that } A\mathbf{x} = \mathbf{0} \text{ is } \mathbf{x} = \mathbf{0} \\ & \text{iff} & \text{the columns of } A \text{ are linearly independent.} \end{array}$$

QED

Matrix invertibility examples

$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ is not square so cannot be invertible.

$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ is square and its columns are linearly independent so it is invertible.

$\left[\begin{array}{c|c|c} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \end{array}\right]$ is square but columns not linearly independent so it is not invertible.

Transpose of invertible matrix is invertible

Theorem: The transpose of an invertible matrix is invertible.

$$A = \left[\begin{array}{c|c|c} \mathbf{v}_1 & \cdots & \mathbf{v}_n \end{array}\right] = \left[\begin{array}{c} \mathbf{a}_1 \\ \hline \vdots \\ \hline \mathbf{a}_n \end{array}\right] \qquad A^T = \left[\begin{array}{c|c|c} \mathbf{a}_1 & \cdots & \mathbf{a}_n \end{array}\right]$$

Proof: Suppose A is invertible. Then A is square and its columns are linearly independent. Let n be the number of columns. Then rank $A = n$.

Because A is square, it has n rows. By Rank Theorem, rows are linearly independent.

Columns of transpose A^T are rows of A, so columns of A^T are linearly independent.

Since A^T is square and columns are linearly independent, A^T is invertible. QED

More matrix invertibility

Earlier we proved: *If A has an inverse A^{-1} then AA^{-1} is identity matrix*

Converse: If BA is identity matrix then A and B are inverses? **Not always true.**

Theorem: *Suppose A and B are square matrices such that BA is an identity matrix $\mathbb{1}$. Then A and B are inverses of each other.*

Proof: To show that A is invertible, need to show its columns are linearly independent.

Let $\mathbf{u}$ be any vector such that $A\mathbf{u} = \mathbf{0}$. Then $B(A\mathbf{u}) = B\mathbf{0} = \mathbf{0}$.
On the other hand, $(BA)\mathbf{u} = \mathbb{1}\mathbf{u} = \mathbf{u}$, so $\mathbf{u} = \mathbf{0}$.

This shows A has an inverse A^{-1}. Now must show $B = A^{-1}$.
We know AA^{-1} is an identity matrix.

$$
\begin{aligned}
BA &= \mathbb{1} && \\
(BA)A^{-1} &= \mathbb{1}A^{-1} && \text{by multiplying on the right by } B^{-1} \\
(BA)A^{-1} &= A^{-1} && \\
B(AA^{-1}) &= A^{-1} && \text{by associativity of matrix-matrix mult} \\
B\,\mathbb{1} &= A^{-1} && \\
B &= A^{-1} && \textit{QED}
\end{aligned}
$$

Representations of vector spaces

Two important ways to represent a vector space:

As the solution set of homogeneous linear system

$$\mathbf{a}_1 \cdot \mathbf{x} = 0, \ldots, \mathbf{a}_m \cdot \mathbf{x} = 0$$

Equivalently,

$$\text{Null} \begin{bmatrix} \mathbf{a}_1 \\ \hline \vdots \\ \hline \mathbf{a}_m \end{bmatrix}$$

As Span $\{\mathbf{b}_1, \ldots, \mathbf{b}_k\}$

Equivalently,

$$\text{Row} \begin{bmatrix} \mathbf{b}_1 \\ \hline \vdots \\ \hline \mathbf{b}_k \end{bmatrix}$$

How to transform between these two representations?

From left to right: Given homogeneous linear system $\mathbf{a}_1 \cdot \mathbf{x} = 0, \ldots, \mathbf{a}_m \cdot \mathbf{x} = 0$, find generators $\mathbf{b}_1, \ldots, \mathbf{b}_k$ for solution set

From right to left:
Given generators $\mathbf{b}_1, \ldots, \mathbf{b}_k$,
find homogeneous linear system $\mathbf{a}_1 \cdot \mathbf{x} = 0, \ldots, \mathbf{a}_m \cdot \mathbf{x} = 0$ whose solution set equals Span $\{\mathbf{b}_1, \ldots, \mathbf{b}_k\}$

Annihilator of a vector space

From left to right: Given system $\mathbf{a}_1 \cdot \mathbf{x} = 0, \ldots, \mathbf{a}_m \cdot \mathbf{x} = 0$, find generators $\mathbf{b}_1, \ldots, \mathbf{b}_k$ for solution set

Solution set is the set of vectors $\mathbf{u}$ such that $\mathbf{a}_1 \cdot \mathbf{u} = 0, \ldots, \mathbf{a}_m \cdot \mathbf{u} = 0$

$$\underbrace{\left[\begin{array}{c} \mathbf{a}_1 \\ \hline \vdots \\ \hline \mathbf{a}_m \end{array}\right]}_{A} \left[\begin{array}{c} \mathbf{x} \end{array}\right] = \left[\begin{array}{c} 0 \\ \vdots \\ 0 \end{array}\right]$$

Equivalent: Given rows of a matrix A, find generators for Null A

$$\begin{array}{c} \text{rows of a matrix } A \\ \downarrow \\ \boxed{\text{Algorithm X}} \\ \downarrow \\ \text{generators for Null } A \end{array}$$

If $\mathbf{u}$ is such a vector then

$$\mathbf{u} \cdot (\alpha_1 \, \mathbf{a}_1 + \cdots + \alpha_m \, \mathbf{a}_m) = 0$$

for any coefficients $\alpha_1, \ldots, \alpha_m$.

Definition: The set of vectors $\mathbf{u}$ such that $\mathbf{u} \cdot \mathbf{v} = 0$ for every vector $\mathbf{v}$ in $\mathcal{V}$ is called the *annihilator* of $\mathcal{V}$. Written as $\mathcal{V}^o$.

Example: The annihilator of Span $\{\mathbf{a}_1, \ldots, \mathbf{a}_m\}$ is the solution set for $\mathbf{a}_1 \cdot \mathbf{x} = 0, \ldots, \mathbf{a}_m \cdot \mathbf{x} = 0$

$$\begin{array}{c} \text{generators for a vector space } \mathcal{V} \\ \downarrow \\ \boxed{\text{Algorithm X}} \\ \downarrow \\ \text{generators for annihilator } \mathcal{V}^o \end{array}$$

Annihilator of a vector space

Definition: For a subspace $\mathcal{V}$ of $\mathbb{F}^n$, the *annihilator* of $\mathcal{V}$, written $\mathcal{V}^o$, is

$$\mathcal{V}^o = \{\mathbf{u} \in \mathbb{F}^n \;:\; \mathbf{u} \cdot \mathbf{v} = 0 \text{ for every vector } \mathbf{v} \in \mathcal{V}\}$$

Example over $\mathbb{R}$: Let $\mathcal{V} = \text{Span}\ \{[1,0,1],[0,1,0]\}$. Then $\mathcal{V}^o = \text{Span}\ \{[1,0,-1]\}$:

- Note that $[1,0,-1] \cdot [1,0,1] = 0$ and $[1,0,-1] \cdot [0,1,0] = 0$.
  Therefore $[1,0,-1] \cdot \mathbf{v} = 0$ for every vector $\mathbf{v}$ in Span $\{[1,0,1],[0,1,0]\}$.
- For any scalar β,
  $$\beta\,[1,0,-1] \cdot \mathbf{v} = \beta\,([1,0,-1] \cdot \mathbf{v}) = 0$$
  for every vector $\mathbf{v}$ in Span $\{[1,0,1],[0,1,0]\}$.
- Which vectors $\mathbf{u}$ satisfy $\mathbf{u} \cdot \mathbf{v} = 0$ for every vector $\mathbf{v}$ in Span $\{[1,0,1],[0,1,0]\}$?
  Only scalar multiples of $[1,0,-1]$.

Example over $GF(2)$: Let $\mathcal{V} = \text{Span}\ \{[1,0,1],[0,1,0]\}$. Then $\mathcal{V}^o = \text{Span}\ \{[1,0,1]\}$:

- Note that $[1,0,1] \cdot [1,0,1] = 0$ (remember $GF(2)$ addition) and $[1,0,1] \cdot [0,1,0] = 0$.
- Therefore $[1,0,1] \cdot \mathbf{v} = 0$ for every vector $\mathbf{v}$ in Span $\{[1,0,1],[0,1,0]\}$.
- Of course $[0,0,0] \cdot \mathbf{v} = 0$ for every vector $\mathbf{v}$ in Span $\{[1,0,1],[0,1,0]\}$.
- $[1,0,1]$ and $[0,0,0]$ are the only such vectors.

Annihilator of a vector space

Example over $\mathbb{R}$: Let $\mathcal{V} = \text{Span } \{[1, 0, 1], [0, 1, 0]\}$. Then $\mathcal{V}^o = \text{Span } \{[1, 0, -1]\}$
$\dim \mathcal{V} + \dim \mathcal{V}^o = 3$

Example over $GF(2)$: Let $\mathcal{V} = \text{Span } \{[1, 0, 1], [0, 1, 0]\}$. Then $\mathcal{V}^o = \text{Span } \{[1, 0, 1]\}$.
$\dim \mathcal{V} + \dim \mathcal{V}^o = 3$

Example over $\mathbb{R}$: Let $\mathcal{V} = \text{Span } \{[1, 0, 1, 0], [0, 1, 0, 1]\}$.
Then $\mathcal{V}^o = \text{Span } \{[1, 0, -1, 0], [0, 1, 0, -1]\}$.
$\dim \mathcal{V} + \dim \mathcal{V}^o = 4$

Annihilator Dimension Theorem: $\dim \mathcal{V} + \dim \mathcal{V}^o = n$

Proof: Let $\mathbf{a}_1, \ldots, \mathbf{a}_m$ be generators for $\mathcal{V}$.

Let $A = \begin{bmatrix} \mathbf{a}_1 \\ \hline \vdots \\ \hline \mathbf{a}_m \end{bmatrix}$

Then $\mathcal{V}^o = \text{Null } A$.

Rank-Nullity Theorem states that

$$\begin{array}{ccccc} \text{rank } A & + & \text{nullity } A & = & n \\ \dim \mathcal{V} & + & \dim \mathcal{V}^o & = & n \end{array}$$

QED

Annihilator of a vector space

Definition: For a subspace $\mathcal{V}$ of $\mathbb{F}^n$, the *annihilator* of $\mathcal{V}$, written $\mathcal{V}^o$, is

$$\mathcal{V}^o = \{\mathbf{u} \in \mathbb{F}^n \; : \; \mathbf{u} \cdot \mathbf{v} = 0 \text{ for every vector } \mathbf{v} \in \mathcal{V}\}$$

$$\begin{array}{ccc} \text{rows of a matrix } A & & \text{generators for a vector space } \mathcal{V} \\ \downarrow & & \downarrow \\ \boxed{\text{Algorithm X}} & = & \boxed{\text{Algorithm X}} \\ \downarrow & & \downarrow \\ \text{generators for Null } A & & \text{generators for annihilator } \mathcal{V}^o \end{array}$$

From left to right: Given system $\mathbf{a}_1 \cdot \mathbf{x} = 0, \ldots, \mathbf{a}_m \cdot \mathbf{x} = 0$, find generators $\mathbf{b}_1, \ldots, \mathbf{b}_k$ for solution set

Algorithm X solves left-to-right problem....

what about right-to-left problem?

Annihilator of a vector space

From left to right: Given system
$\mathbf{a}_1 \cdot \mathbf{x} = 0, \ldots, \mathbf{a}_m \cdot \mathbf{x} = 0,$
find generators $\mathbf{b}_1, \ldots, \mathbf{b}_k$ for solution set

$$\text{generators for a vector space } \mathcal{V} \;\downarrow\; \boxed{\text{Algorithm X}} \;\downarrow\; \text{generators for annihilator } \mathcal{V}^o$$

What happens if we apply Algorithm X to generators for annihilator $\mathcal{V}^o$?

$$\text{generators for annihilator } \mathcal{V}^o \;\downarrow\; \boxed{\text{Algorithm X}} \;\downarrow\; \text{generators for annihilator of annihilator } (\mathcal{V}^o)^o$$

From right to left: Given generators $\mathbf{b}_1, \ldots, \mathbf{b}_k$, find system $\mathbf{a}_1 \cdot \mathbf{x} = 0, \ldots, \mathbf{a}_m \cdot \mathbf{x} = 0$ whose solution set equals Span $\{\mathbf{b}_1, \ldots, \mathbf{b}_k\}$

$$\text{generators for annihilator } \mathcal{V}^o \;\downarrow\; \boxed{\text{Algorithm Y}} \;\downarrow\; \text{generators for original space } \mathcal{V}$$

Theorem: $(\mathcal{V}^o)^o = \mathcal{V}$ (The annihilator of the annihilator is the original space.)

Theorem shows:

$$\text{Algorithm X} = \text{Algorithm Y}$$

We still must prove the Theorem...

Annihilator

Theorem: $(\mathcal{V}^o)^o = \mathcal{V}$ (The annihilator of the annihilator is the original space.)

Proof:
Let $\mathbf{a}_1, \ldots, \mathbf{a}_m$ be a basis for $\mathcal{V}$. Let $\mathbf{b}_1, \ldots, \mathbf{b}_k$ be a basis for $\mathcal{V}^o$.
Since $\mathbf{b}_1 \cdot \mathbf{v} = 0$ for every vector $\mathbf{v}$ in $\mathcal{V}$,

$$\mathbf{b}_1 \cdot \mathbf{a}_1 = 0, \mathbf{b}_1 \cdot \mathbf{a}_2 = 0, \ldots, \mathbf{b}_1 \cdot \mathbf{a}_m = 0$$

Similarly $\mathbf{b}_i \cdot \mathbf{a}_1 = 0, \mathbf{b}_i \cdot \mathbf{a}_2 = 0, \ldots, \mathbf{b}_i \cdot \mathbf{a}_m = 0$ for $i = 1, 2, \ldots, k$.

Reorganizing,

$$\mathbf{a}_1 \cdot \mathbf{b}_1 = 0, \mathbf{a}_1 \cdot \mathbf{b}_2 = 0, \ldots, \mathbf{a}_1 \cdot \mathbf{b}_k = 0$$

which implies that $\mathbf{a}_1 \cdot \mathbf{u} = 0$ for every vector $\mathbf{u}$ in $\underbrace{\text{Span }\{\mathbf{b}_1, \ldots, \mathbf{b}_k\}}_{\mathcal{V}^o}$

This shows $\mathbf{a}_1$ is in $(\mathcal{V}^o)^o$. Similarly $\mathbf{a}_2$ is in $(\mathcal{V}^o)^o$, $\mathbf{a}_3$ is in $(\mathcal{V}^o)^o$, ..., $\mathbf{a}_m$ is in $(\mathcal{V}^o)^o$.

Therefore every vector in Span $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m\}$ is in $(V^o)^o$.

Thus $\underbrace{\text{Span }\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m\}}_{\mathcal{V}}$ is a subspace of $(\mathcal{V}^o)^o$.

To show that these are equal, we must show that $\dim \mathcal{V} = \dim(\mathcal{V}^o)^o$.

By Annihilator Dimension Theorem, $\dim \mathcal{V} + \dim \mathcal{V}^o = n$

Annihilator

Theorem: $(\mathcal{V}^o)^o = \mathcal{V}$ (The annihilator of the annihilator is the original space.)

Proof:
Reorganizing,

$$\mathbf{a}_1 \cdot \mathbf{b}_1 = 0, \mathbf{a}_1 \cdot \mathbf{b}_2 = 0, \ldots, \mathbf{a}_1 \cdot \mathbf{b}_k = 0$$

which implies that $\mathbf{a}_1 \cdot \mathbf{u} = 0$ for every vector $\mathbf{u}$ in $\underbrace{\text{Span } \{\mathbf{b}_1, \ldots, \mathbf{b}_k\}}_{\mathcal{V}^o}$

This shows $\mathbf{a}_1$ is in $(\mathcal{V}^o)^o$. Similarly $\mathbf{a}_2$ is in $(\mathcal{V}^o)^o$, $\mathbf{a}_3$ is in $(\mathcal{V}^o)^o$, ..., $\mathbf{a}_m$ is in $(\mathcal{V}^o)^o$.

Therefore every vector in Span $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m\}$ is in $(V^o)^o$.

Thus $\underbrace{\text{Span } \{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m\}}_{\mathcal{V}}$ is a subspace of $(\mathcal{V}^o)^o$.

To show that these are equal, we must show that $\dim \mathcal{V} = \dim(\mathcal{V}^o)^o$.

By Annihilator Dimension Theorem, $\dim \mathcal{V} + \dim \mathcal{V}^o = n$.

By Annihilator Dimension Theorem applied to $\mathcal{V}^o$, $\dim \mathcal{V}^o + \dim(\mathcal{V}^o)^o = n$.

Together these equations show $\dim \mathcal{V} = \dim(\mathcal{V}^o)^o$. QED